BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 71(4), 2023, Article number: e145938
DOI: 10.24425/bpasts.2023.145938

Existence, uniqueness and parameter perturbation analysis results of a fractional integro-differential boundary problem

Nan ZHANG[1], Lingling ZHANG[1]*, Mercy NGUNGU[2], Adejimi ADENIJI[3], and Emmanuel ADDAI[1]

[1] College of Mathematics, Taiyuan University of Technology, 030024, TaiYuan, Shanxi, China
[2] Human Sciences Research Council (HSRC), South Africa
[3] Tshwane university of Technology, South Africa

Abstract. In the formulation, the existence, uniqueness and stability of solutions and parameter perturbation analysis to Riemann-Liouville fractional differential equations with integro-differential boundary conditions are discussed by the properties of Green's function and cone theory. First, some theorems have been established from standard fixed point theorems in a proper Banach space to guarantee the existence and uniqueness of positive solution. Moreover, we discuss the Hyers-Ulam stability and parameter perturbation analysis, which examines the stability of solutions in the presence of small changes in the equation main parameters, that is, the derivative order η, the integral order β of the boundary condition, the boundary parameter ξ, and the boundary value τ. As an application, we present a concrete example to demonstrate the accuracy and usefulness of the proposed work. By using numerical simulation, we obtain the figure of unique solution and change trend figure of the unique solution with small disturbances to occur in different kinds of parameters.

Key words: existence and uniqueness; stability analysis; parameter perturbation; fractional differential equation; integro-differential boundary condition.

1. INTRODUCTION

It is shown through experimentation that the fractional differential equations (FDEs) do share memory and genetic characteristics exhibited by the process of complex system modeling. Due to the modeling capabilities of FDEs in engineering and science field, more and more scholars pay attention to the study of FDEs [1–9]. In recent years, research on the initial and boundary value problems of FDEs has made rapid progress, especially in the study of the existence, uniqueness or multiplicity of positive solutions to fractional boundary value problems. By using various analytical tools in nonlinear functional analysis, such as Schauder fixed point theorem [10–12], contraction mapping principle [13–16], topological degree theory [17, 18], monotonic iteration technique [19–21] among others, a large number of novel and meaningful conclusions have emerged. Taking the famous Schauder fixed point theorem and contraction mapping principle as examples, the literature [10–16] and their corresponding references have studied several different types of boundary value problems for FDEs based on these two theorems. For instance, [10] considered the following problems

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t) = f(t,x(t)) + {}^cD^{\alpha-1}_{0^+}g(t,x(t)), \\ \quad x(0) = \theta_1 > 0, \quad x'(0) = \theta_2 > 0, \end{cases}$$

where ${}^cD^{\alpha}_{0^+}$ is Caputo fractional derivative, $1 < \alpha \le 2$. The existence of positive solution was proved using Schauder's fixed point theorem, and the existence of a unique positive solution was shown using the contraction mapping principle. In [12], the authors obtained the existence and uniqueness of solutions using the above two theorems,

$$\begin{cases} {}^cD^{q}_{0^+,\psi}x(t) = f(t,x(t)), \quad t \in [0,T], \ 1 < q \le 2, \\ x(0) - \delta_\psi x(0) = I^{\alpha}_{0^+,\psi}g(\sigma,x(\sigma)), \\ x(T) + \delta_\psi x(T) = I^{\beta}_{0^+,\psi}h(\eta,x(\eta)). \end{cases}$$

In fact, from the aforementioned literature, the Banach space $E = \{x | x \in C[0,1]\}$ and cone $P = \{x \in E | x(t) \ge 0, t \in [0,1]\}$ required for Schauder's fixed point theorem and contraction mapping principle are the most classical, and the proof process is relatively simple.

The ability to maintain a control system stability in the presence of parameter perturbations is crucial for system design, and as a result, extensive study has been done in this field to study the necessary conditions for stability. We discovered numerous methods in the literature for examining stability of the problem, including exponential, Lyapunov, asymptotic, and Hyers-Ulam stability, among others [22–25]. In recent years, the works in [22–28] conducted in-depth research on Hyers-Ulam stability of solutions to FDEs because Hyers-Ulam stability is a chain among exact and numerical solutions. In addition, we found that the perturbation analysis of the solution is not extensive, and the study findings are not particularly substantial.

*e-mail: tyutzll@126.com

Manuscript submitted 2022-11-26, revised 2023-04-24, initially accepted for publication 2023-05-16, published in August 2023.

Oriented by the above discussion, we study the following integro-differential boundary value problem of FDEs:

$$\begin{cases} D_{0+}^{\eta}x(t)+f(t,x(t),D_{0+}^{\alpha}x(t))=0, \quad t\in[0,1], \\ x^{(i)}(0)=0, \quad i=0,1,\cdots,n-2, \\ D_{0+}^{\alpha}x(1)=\tau I_{0+}^{\beta}x(\xi)=\tau\int\limits_{0}^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}x(s)\,\mathrm{d}s, \end{cases} \tag{1}$$

where D_{0+}^{η}, D_{0+}^{α} are Riemann-Liouville (R-L) fractional derivative, I_{0+}^{β} is R-L fractional integral, $0<\alpha<n-1<\eta\le n$ $(n>1)$, $\eta-\alpha-1>0$, $\beta,\ \tau>0$, $0<\xi\le 1$, $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous. As the nonlinear term contains derivative term, the classic and common Banach space and cone in Schauder fixed point theorem and contraction mapping principle are no longer applicable. In this article, we constructed a suitable Banach space and cone with a new norm to gain the existence and uniqueness of positive solution. In addition, Hyers-Ulam stability and parameter perturbation analysis are discussed.

It is worth noting the following points. (i) A proper choice of Banach space allows the nonlinear term of the equation to contain derivative term. (ii) We explore how the solution depends on parameters under some small perturbations of main parameters in the equation based on the finding that the solutions for a class of fractional integro-differential equations exist, are unique, and exhibit Hyers-Ulam stability. In fact, our research demonstrates that the solution is continuously dependent on these key variables. (iii) Through numerical simulation, we obtain graphical presentation of approximate unique solution as the main parameters in equation are slightly disturbed which verify the effectiveness of our theoretical conclusions.

2. PRELIMINARIES

To better serve later contents, we revisit some definitions and lemmas.

Definition 1 [1]. The R-L fractional derivative of $h\colon(0,+\infty)\to R$ is

$$D_{0+}^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_{0}^{t}(t-s)^{n-\alpha-1}h(s)\,\mathrm{d}s.$$

The R-L fractional integral of h is

$$I_{0+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-s)^{\alpha-1}h(s)\,\mathrm{d}s,$$

where $n=[\alpha]+1$, $[\alpha]$ means the integer part of number α, provided the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 1 [2]. If $h\in C(0,1)\cup L^1(0,1)$, $D_{0+}^{\alpha}h\in C(0,1)\cup L^1(0,1)$, $\alpha>0$, then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}h(t)=h(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}, \quad \alpha>0,$$

where $c_i\in R$, $i=1,2,\cdots,n$ $(n=[\alpha]+1)$.

Lemma 2. If $h\in C[0,1]$, $\Gamma(\eta+\beta)>\tau\Gamma(\eta-\alpha)\xi^{\eta+\beta-1}$, the following equation

$$\begin{cases} D_{0+}^{\eta}x(t)+\sigma(t)=0, \\ x^{(i)}(0)=0, \quad i=0,1,\cdots,n-2, \\ D_{0+}^{\alpha}x(1)=\tau I_{0+}^{\beta}x(\xi)=\tau\int\limits_{0}^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}x(s)\,\mathrm{d}s, \end{cases} \tag{2}$$

has a unique solution $x(t)=\int\limits_{0}^{1}G(t,s)\sigma(s)\,\mathrm{d}s$, where

$$G(t,s)=\begin{cases} \dfrac{\Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1}-M(t-s)^{\eta-1}}{M\Gamma(\eta)} \\ \quad -\dfrac{\tau\Gamma(\eta-\alpha)t^{\eta-1}(\xi-s)^{\eta+\beta-1}}{M}, \\ \qquad 0\le s\le t\le 1, \quad s\le\xi, \\ \dfrac{\Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1}-M(t-s)^{\eta-1}}{M\Gamma(\eta)}, \\ \qquad 0\le\xi\le s\le t\le 1, \\ \dfrac{\Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1}}{M\Gamma(\eta)}, \\ \quad -\dfrac{\tau\Gamma(\xi)\Gamma(\eta-\alpha)t^{\eta-1}(\xi-s)^{\eta+\beta-1}}{M\Gamma(\eta)}, \\ \qquad 0\le t\le s\le\xi\le 1, \\ \dfrac{\Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1}}{M\Gamma(\eta)}, \\ \qquad 0\le t\le s\le 1, \quad \xi\le s. \end{cases}$$

and $M=\Gamma(\eta)\Gamma(\eta+\beta)-\tau\Gamma(\eta)\Gamma(\eta-\alpha)\xi^{\eta+\beta-1}$.

Proof. Integrating η times to the first formula of equation (2), then by Lemma 2, we can obtain

$$x(t)=-I_{0+}^{\eta}\sigma(t)+c_1t^{\eta-1}+c_2t^{\eta-2}+\cdots+c_nt^{\eta-n}.$$

From $x^{(i)}(0)=0$ $(i=0,1,\cdots,n-2)$, we see that $c_n=c_{n-1}=\cdots=c_2=0$. Thus, the solution of equation (2) is

$$x(t)=-\int\limits_{0}^{t}\frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\sigma(s)\,\mathrm{d}s+c_1t^{\eta-1}. \tag{3}$$

According to $D_{0+}^{\alpha}x(1)=\tau I_{0+}^{\beta}x(\xi)$, we conclude

$$c_1=\frac{\Gamma(\eta+\beta)}{M}\int\limits_{0}^{1}(1-s)^{\eta-\alpha-1}\sigma(s)\,\mathrm{d}s - \frac{\Gamma(\eta-\alpha)}{M}\int\limits_{0}^{\xi}\tau(\xi-s)^{\eta+\beta-1}\sigma(s)\,\mathrm{d}s. \tag{4}$$

Substituting (4) in (3), there is

$$\begin{aligned} x(t) &= -\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\sigma(s)\,\mathrm{d}s + c_1 t^{\eta-1} \\ &= -\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\sigma(s)\,\mathrm{d}s + \frac{\Gamma(\eta+\beta)t^{\eta-1}}{M}\int_0^1 \\ &\quad \cdot (1-s)^{\eta-\alpha-1}\sigma(s)\,\mathrm{d}s - \frac{\Gamma(\eta-\alpha)t^{\eta-1}}{M} \\ &\quad \cdot \int_0^\xi \tau(\xi-s)^{\eta+\beta-1}\sigma(s)\,\mathrm{d}s = \int_0^1 G(t,s)\sigma(s)\,\mathrm{d}s. \end{aligned}$$

□

Lemma 3. The following properties are established:

(i) For any $t,s \in [0,1]$, $G(t,s), D^\alpha_{0^+}G(t,s)$ are continuous.

(ii) For any $t,s \in [0,1]$, it gives

$$0 \le l_1(s) \le G(t,s) \le \frac{\Gamma(\eta+\beta)}{M} := M_1\,,$$
$$0 \le l_2(s) \le D^\alpha_{0^+}G(t,s) \le \frac{\Gamma(\eta)\Gamma(\eta+\beta)}{M\Gamma(\eta-\alpha)} := M_2\,.$$

Proof. From the definition of $G(t,s)$ and M, when $0 \le s \le t \le 1, s \le \xi$, we observe that

$$\begin{aligned} M\Gamma(\eta)G(t,s) &= \Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1} - M(t-s)^{\eta-1} \\ &\quad - \tau\Gamma(\eta)\Gamma(\eta-\alpha)t^{\eta-1}(\xi-s)^{\eta+\beta-1} \\ &\le \Gamma(\eta)\Gamma(\eta+\beta), \end{aligned}$$

and

$$\begin{aligned} M\Gamma(\eta)G(t,s) &= \Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1} \\ &\quad - \Gamma(\eta)\Gamma(\eta+\beta)(t-s)^{\eta-1} \\ &\quad + \tau\Gamma(\eta)\Gamma(\eta-\alpha)\xi^{\eta+\beta-1}(t-s)^{\eta-1} \\ &\quad - \tau\Gamma(\eta)\Gamma(\eta-\alpha)t^{\eta-1}(\xi-s)^{\eta+\beta-1} \\ &\ge \tau\Gamma(\eta)\Gamma(\eta-\alpha)t^{\eta-1}[\xi^{\eta+\beta-1}(1-s)^{\eta-\alpha-1} \\ &\quad - (\xi-s)^{\eta+\beta-1}] \\ &\ge \tau\Gamma(\eta)\Gamma(\eta-\alpha)\xi^{\eta+\beta-1}t^{\eta-1}(1-s)^{\eta-\alpha-1} \\ &\quad \cdot \left[1-(1-s)^{\alpha+\beta}\right] := \frac{l_1(s)}{M\Gamma(\eta)} \ge 0. \end{aligned}$$

From the definition of $D^\alpha_{0^+}G(t,s)$, and M, when $0 \le s \le t \le 1, s \le \xi$, we observe that

$$\begin{aligned} &M\Gamma(\eta-\alpha)D^\alpha_{0^+}G(t,s) = \Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-\alpha-1}(1-s)^{\eta-\alpha-1} \\ &\quad - M(t-s)^{\eta-\alpha-1} - \tau\Gamma(\eta)\Gamma(\eta-\alpha)t^{\eta-\alpha-1}(\xi-s)^{\eta+\beta-1} \\ &\quad \le \Gamma(\eta)\Gamma(\eta+\beta), \end{aligned}$$

and

$$\begin{aligned} M\Gamma(\eta-\alpha)D^\alpha_{0^+}G(t,s) &\ge \tau\Gamma(\eta)\Gamma(\eta-\alpha)\xi^{\eta+\beta-1} \\ &\quad \cdot [t^{\eta-\alpha-1}(1-s)^{\eta-\alpha-1} - (t-s)^{\eta-\alpha-1}] \\ &\quad + \tau\Gamma(\eta)\Gamma(\eta-\alpha)\xi^{\eta+\beta-1}(t-s)^{\eta-\alpha-1} \\ &\quad - \tau\Gamma(\eta)\Gamma(\eta-\alpha)t^{\eta-\alpha-1}(\xi-s)^{\eta+\beta-1} \\ &\ge \tau\Gamma(\eta)\Gamma(\eta-\alpha)t^{\eta-\alpha-1}\xi^{\eta+\beta-1}(1-s)^{\eta-\alpha-1} \\ &\quad \cdot [1-(1-s)^{\alpha+\beta}] := \frac{l_2(s)}{M\Gamma(\eta)} \ge 0, \end{aligned}$$

In the same way, we discuss the case of $M\Gamma(\eta)G(t,s)$ and $M\Gamma(\eta-\alpha)D^\alpha_{0^+}G(t,s)$ when $0 \le \xi \le s \le t \le 1$, $0 \le t \le s \le \xi \le 1$, and $0 \le t \le s \le 1$, $\xi \le s$, respectively, we can obtain that (ii) holds. □

3. EXISTENCE, UNIQUENESS AND HYERS-ULAM STABILITY OF THE SOLUTION

In this section, we use lemmas to obtain existence, uniqueness and Hyers-Ulam stability of solutions for equation (1).

Firstly, set $E = \{x | x \in C[0,1], D^\alpha_{0^+}x(t) \in C[0,1]\}$ with norm $\|x\| = \max\{\max_{0\le t\le 1}|x(t)|, \max_{0\le t\le 1}|D^\alpha_{0^+}x(t)|\}$, $P = \{x \in E |\, x(t) \ge 0, D^\alpha_{0^+}x(t) \ge 0\}$. Then, E is a Banach space, $P \subset E$ is a cone. Define a partial order $x \le y$ if $x(t) \le y(t)$, $D^\alpha_{0^+}x(t) \le D^\alpha_{0^+}y(t)$, $t \in [0,1]$.

Lemma 4. Let $f : [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ be continuous. Define an operator $A : P \to P$ by

$$Ax(t) = \int_0^1 G(t,s)f(s,x(s),D^\alpha_{0^+}x(s))\,\mathrm{d}s,$$

then the operator A is completely continuous.

Proof. Step 1: $A\colon P \to P$ is continuous. Basing on the definition of A, f and $G(t,s)$, we can easily get that $A(P) \subset P$. Let u_n, $u \in P$, there is $u_n \to u$ as $n \to \infty$. By the Lebesgue dominated convergence theorem, we have $A\colon P \to P$ is continuous.

Step 2: For any $u \in U$ (U is a bounded subset of P), by the Lemma 3, there is

$$|Au(t)| = \max_{t\in[0,1]}\left|\int_0^1 G(t,s)f(s,u(s),D^\alpha_{0^+}u(s))\,\mathrm{d}s\right| \le NM_1,$$
$$|D^\alpha_{0^+}Au(t)| = \max_{t\in[0,1]}\left|\int_0^1 D^\alpha_{0^+}G(t,s)f(s,u(s),D^\alpha_{0^+}u(s))\,\mathrm{d}s\right| \le NM_2,$$

which implies that $\|Au\| = NM_1 + NM_2 := L$, where $N = \max_{(t,u)\in[0,1]\times U} f(s,u(s),D^\alpha_{0^+}u(s))$.

Step 3: Due to $G(t,s)$, $D^\alpha_{0^+}G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for any $\varepsilon > 0$, $\exists\delta > 0$, $0 \le t_2 - t_1 \le \delta$, $t_1,t_2 \in [0,1]$, such that

$$\left|G(t_2,s) - G(t_1,s)\right| < \frac{\varepsilon}{N}, \quad \left|D^\alpha_{0^+}G(t_2,s) - D^\alpha_{0^+}G(t_1,s)\right| < \frac{\varepsilon}{N}.$$

For any $u \in U$, $t_1,\ t_2 \in [0,1]$, $0 \le t_2 - t_1 \le \delta$, we get that

$$\left|Au(t_2) - Au(t_1)\right| \le \int_0^1 \left|G(t_2,s) - G(t_1,s)\right| f(s,u(s),D^{\alpha}_{0^+}u(s))\,\mathrm{d}s \le \frac{\varepsilon}{N}\cdot N = \varepsilon,$$

$$\left|D^{\alpha}_{0^+}Au(t_2) - D^{\alpha}_{0^+}Au(t_1)\right| \le \int_0^1 \left|D^{\alpha}_{0^+}G(t_2,s) - D^{\alpha}_{0^+}G(t_1,s)\right| \cdot f(s,u(s),D^{\alpha}_{0^+}u(s))\,\mathrm{d}s \le \frac{\varepsilon}{N}\cdot N = \varepsilon.$$

i.e., $\{A(u)\}$ is equicontinuous. Hence, by Arzela-Ascoli theorem, $A\colon P \to P$ is completely continuous. □

Theorem 1. Let $f\colon [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ be continuous. Assume that

(H_1) there exist nonnegative constants $L_i(t) \in L^1[0,1]\bigcap C[0,1]$ $(i = 0,1,2)$ such that

$$|f(t,x,y)| \le L_0(t) + L_1(t)|x| + L_2(t)|y|;$$

(H_2)
$$M_1 \int_0^1 (L_1(s)+L_2(s))\,\mathrm{d}s < \frac{1}{2},$$
$$M_2 \int_0^1 (L_1(s)+L_2(s))\,\mathrm{d}s < \frac{1}{2};$$

Then equation (1) has at least one solution on P.

Proof. Take $r_{11} = \max\left\{2M_1\int_0^1 L_0(s)\,\mathrm{d}s, 2M_2\int_0^1 L_0(s)\,\mathrm{d}s\right\}$, let $r_1 > r_{11}$, and $\Omega = \{u|u \in E, \|u\| \le r_1\}$. Then, Ω is a nonempty bounded closed convex sets of E. From (H_1), for any $u \in \Omega$, we have

$$\begin{aligned}|Au(t)| &= \left|\int_0^1 G(t,s)f(s,u(s),D^{\alpha}_{0^+}u(s))\,\mathrm{d}s\right| \\ &\le \frac{\Gamma(\eta+\beta)}{M}\left|\int_0^1 f(s,u(s),D^{\alpha}_{0^+}u(s))\,\mathrm{d}s\right| \\ &\le M_1\int_0^1 \Big(L_0(s)+L_1(s)|u(s)|+L_2(s)|D^{\alpha}_{0^+}u(s)|\Big)\,\mathrm{d}s \\ &\le M_1\int_0^1 L_0(s)\,\mathrm{d}s + M_1\int_0^1 (L_1(s)+L_2(s))\,\mathrm{d}s\cdot\|u\|,\end{aligned}$$

$$\begin{aligned}|D^{\alpha}Au(t)| &= \left|\int_0^1 G(t-s)f(s,u(s),D^{\alpha}_{0^+}u(s))\,\mathrm{d}s\right| \\ &\le \frac{\Gamma(\eta)\Gamma(\eta+\beta)}{M\Gamma(\eta-\alpha)}\left|\int_0^1 f(s,u(s),D^{\alpha}_{0^+}u(s))\,\mathrm{d}s\right| \\ &\le M_2\int_0^1 \Big(L_0(s)+L_1(s)|u(s)|+L_2(s)|D^{\alpha}_{0^+}u(s)|\Big)\,\mathrm{d}s \\ &\le M_2\int_0^1 L_0(s)\,\mathrm{d}s + M_2\int_0^1 (L_1(s)+L_2(s))\,\mathrm{d}s\cdot\|u\|.\end{aligned}$$

From (H_2), there is

$$|Au(t)| \le \frac{r_{11}}{2} + \frac{1}{2}\|u\| \le r_1, \quad |(D^{\alpha}Au)(t)| \le \frac{r_{11}}{2} + \frac{1}{2}\|u\| \le r_1,$$

i.e., $\|Au\| \le r_1$. Hence, $A(\Omega) \subset \Omega$. Due to the completely continuity of A, and by Schauder fixed point theorem, A has at least one fixed point on Ω. Then, we get that equation (1) has at least one solution on P. □

Corollary 1. As the condition (H_1) turn into (H_1') there exist nonnegative constants $L_i(t) \in L^1[0,1]\bigcap C[0,1]$ $(i = 0,1,2)$ such that

$$|f(t,x,y)| \le L_0(t) + L_1(t)|x|^k + L_2(t)|y|^k,$$

then equation (1) has at least one solution on P.

Remark 1. If $L_i(t)$ $(i = 0,1,2)$ are some constants, or $L_0(t) = 0$ in (H_1'), (H_1), then equation (1) has at least one solution on E.

Theorem 2. Let $f : [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ be continuous. Assume that

(H_3) there exist nonnegative constants $L_i(t) \in L^1[0,1]\bigcap C[0,1]$ $(i = 3,4)$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le L_3(t)|x-\bar{x}| + L_4(t)|y-\bar{y}|;$$

(H_4)
$$\max\{M_1,M_2\}\cdot\int_0^1 \big(L_3(s)+L_4(s)\big)\,\mathrm{d}s < 1.$$

Then equation (1) has a unique solution $u \in P$.

Proof. For any $u_2, u_1 \in E$, by (H_3), we observe that

$$\begin{aligned}|Au_2(t) - Au_1(t)| &\le \int_0^1 G(t,s)\Big|f(s,u_2(s),D^{\alpha}_{0^+}u_2(s)) - f(s,u_1(s),D^{\alpha}_{0^+}u_1(s))\Big|\,\mathrm{d}s \\ &\le M_1\int_0^1 \big(L_3(s)|u(s)|+L_4(s)|D^{\alpha}_{0^+}u(s)|\big)\,\mathrm{d}s \\ &\le M_1\int_0^1 \big(L_3(s)+L_4(s)\big)\,\mathrm{d}s\cdot\|u_2-u_1\|,\end{aligned}$$

$$|D_{0+}^{\alpha}Au_2(t)-D_{0+}^{\alpha}Au_1(t)|\le\int_0^1 D_{0+}^{\alpha}G(t,s)$$
$$\cdot\left|f(s,u_2(s),D_{0+}^{\alpha}u_2(s))-f(s,u_1(s),D_{0+}^{\alpha}u_1(s))\right|\mathrm{d}s$$
$$\le M_2\int_0^1\big(L_3(s)|u(s)|+L_4(s)|D_{0+}^{\alpha}u(s)|\big)\,\mathrm{d}s$$
$$\le M_2\int_0^1\big(L_3(s)+L_4(s)\big)\,\mathrm{d}s\cdot\|u_2-u_1\|.$$

By the condition (H_4), we obtain

$$\|Au_2-Au_1\|\le\max\{M_1,M_2\}\cdot\int_0^1(L_3(s)+L_4(s))\,\mathrm{d}s$$
$$\cdot\|u_2-u_1\|\le\|u_2-u_1\|,$$

which show that A is a contraction mapping. By contraction mapping principle, equation (1) has a unique solution $u\in P$. □

Definition 2. Assume that there exist positive constant k_1, satisfying $\forall p_1>0$, if

$$\left|\psi(t)-\int_0^1 G(t,s)f(s,\psi(s),D_{0+}^{\alpha}\psi(s))\,\mathrm{d}s\right|\le p_1,$$

there exists δ, meeting

$$\delta(t)=\int_0^1 G(t,s)f(s,\delta(s),D_{0+}^{\alpha}\delta(s))\,\mathrm{d}s,\tag{5}$$

such that

$$\|\psi-\delta\|\le k_1p_1,$$

then, equation (1) is Hyers-Ulam stable.

Theorem 3. Let δ be the unique solution of equation (1) and δ satisfies (5). If (H_3) hold, then equation (1) is Hyers-Ulam stable.

Proof. From Lemma 3, we obtain

$$\left|\psi(t)-\delta(t)\right|\le\int_0^1 G(t,s)\Big|f(s,\psi(s),D_{0+}^{\alpha}\psi(s))$$
$$-f(s,\delta(s),D_{0+}^{\alpha}\delta(s))\Big|\,\mathrm{d}s$$
$$\le M_1\int_0^1\big(L_3(s)+L_4(s)\big)\,\mathrm{d}s\cdot\|\psi-\delta\|\le k_1p_1,$$

$$\left|D_{0+}^{\alpha}\psi(t)-D_{0+}^{\alpha}\delta(t)\right|\le\int_0^1 D_{0+}^{\alpha}G(t,s)\Big|f(s,\psi(s),D_{0+}^{\alpha}\psi(s))$$
$$-f(s,\delta(s),D_{0+}^{\alpha}\delta(s))\Big|\,\mathrm{d}s$$
$$\le M_2\int_0^1\big(L_3(s)+L_4(s)\big)\,\mathrm{d}s\cdot\|\psi-\delta\|\le k_1p_1,$$

where $k_1=(M_1+M_2)\int_0^1\big(L_3(s)+L_4(s)\big)\,\mathrm{d}s$. Then, from Definition 2, equation (1) is Hyers-Ulam stable. □

4. INFLUENCE OF PARAMETERS

In this section, we discuss the stability of the solution to equation (1) when there are some small perturbations in the parameters of equation.

As the order of the differential derivative η changes slightly, the stability of the solution is as follows.

Theorem 4. Let x be the solution of equation (1), and let y be the solution of the following problem:

$$\begin{cases}D_{0+}^{\widetilde{\eta}}y(t)+f(t,y(t),D_{0+}^{\alpha}y(t))=0,\\ y^{(i)}(0)=0,\quad i=0,1,\cdots,n-2,\\ D_{0+}^{\alpha}y(1)=\tau I_{0+}^{\beta}y(\xi)=\tau\displaystyle\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}y(s)\,\mathrm{d}s,\end{cases}$$

where $|\eta-\widetilde{\eta}|<\varepsilon$, ε is an any small constant. Then,

$$\|x-y\|\le\frac{M_3+M_4}{1-(M_1+M_2)(L_3+L_4)}\varepsilon.$$

Proof. For any $t\in[0,1]$, there is

$$|x(t)-y(t)|\le\int_0^t\Big|\frac{(t-s)^{\eta-1}}{\Gamma(\eta)}f(s,x(s),D_{0+}^{\alpha}x(s))$$
$$-\frac{(t-s)^{\widetilde{\eta}-1}}{\Gamma(\widetilde{\eta})}f(s,y(s),D_{0+}^{\alpha}y(s))\Big|\,\mathrm{d}s$$
$$+\frac{1}{M}\int_0^1\Big|\Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1}f(s,x(s),D_{0+}^{\alpha}x(s))$$
$$-\Gamma(\widetilde{\eta}+\beta)t^{\widetilde{\eta}-1}(1-s)^{\widetilde{\eta}-\alpha-1}f(s,y(s),D_{0+}^{\alpha}y(s))\Big|\,\mathrm{d}s$$
$$+\frac{\tau}{M}\int_0^{\xi}\Big|\Gamma(\eta-\alpha)t^{\eta-1}(\xi-s)^{\eta+\beta-1}f(s,x(s),D_{0+}^{\alpha}x(s))$$
$$-\Gamma(\widetilde{\eta}-\alpha)t^{\widetilde{\eta}+\beta-1}(\xi-s)^{\widetilde{\eta}-1}f(s,y(s),D_{0+}^{\alpha}y(s))\Big|\,\mathrm{d}s$$
$$:=K_1+K_2+K_3.$$

Using the mean-value theorem we obtain

$$K_1 \leq \int_0^t \left| \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} f(s,x(s),D_{0^+}^{\alpha}x(s)) - \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} \cdot f(s,y(s),D_{0^+}^{\alpha}y(s)) \right| + \left| \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} f(s,y(s),D_{0^+}^{\alpha}y(s)) - \frac{(t-s)^{\widetilde{\eta}-1}}{\Gamma(\widetilde{\eta})} f(s,y(s),D_{0^+}^{\alpha}y(s)) \right| \mathrm{d}s \leq \frac{L_3+L_4}{\Gamma(\eta+1)} \|x-y\| + C_1 \varepsilon \|f\|,$$

where

$$C_1 = \max_{x\in[\eta,\widetilde{\eta}]} \left\{ \left(\frac{t^{x-1}}{\Gamma(x)} \right)', \ 0<t<1 \right\}.$$

Similarly, one can see that

$$K_2 = \frac{1}{M}\int_0^1 \Big| \Gamma(\eta+\beta)t^{\eta-1}(1-s)^{\eta-\alpha-1} f(s,x(s),D_{0^+}^{\alpha}x(s)) - \Gamma(\widetilde{\eta}+\beta)t^{\widetilde{\eta}-1}(1-s)^{\widetilde{\eta}-\alpha-1} f(s,y(s),D_{0^+}^{\alpha}y(s)) \Big| \mathrm{d}s \leq \frac{\Gamma(\eta+\beta)(L_3+L_4)}{M}\|x-y\| + \frac{C_2\|f\|}{M}\varepsilon,$$

$$K_3 = \frac{\tau}{M}\int_0^{\xi} \Big| \Gamma(\eta-\alpha)t^{\eta-1}(\xi-s)^{\eta+\beta-1} f(s,x(s),D_{0^+}^{\alpha}x(s)) - \Gamma(\widetilde{\eta}-\alpha)t^{\widetilde{\eta}-1}(\xi-s)^{\widetilde{\eta}+\beta-1} f(s,y(s),D_{0^+}^{\alpha}y(s)) \Big| \mathrm{d}s \leq \frac{\tau\Gamma(\eta-\alpha)(L_3+L_4)}{M}\|x-y\| + \frac{\tau C_3\|f\|}{M}\varepsilon,$$

where

$$C_2 = \max_{x\in[\eta,\widetilde{\eta}]} \left\{ \left(\Gamma(x+\beta)t^{x-1}(1-s)^{x-\alpha-1} \right)', \ 0<t, \ s<1 \right\},$$

$$C_3 = \max_{x\in[\eta,\widetilde{\eta}]} \left\{ \left(\Gamma(x-\alpha)t^{x-1}(\xi-s)^{x+\beta-1} \right)', \ 0<t, \ \xi, s<1, \ s<\xi \right\}.$$

Thus, we shall get

$$|x(t)-y(t)| \leq \left[\frac{1}{\Gamma(\eta+1)} + \frac{\Gamma(\eta+\beta)}{M} + \frac{\tau\Gamma(\eta-\alpha)}{M} \right](L_3+L_4) \cdot \|x-y\| + \left[C_1\|f\| + \frac{C_2\|f\|}{M} + \frac{\tau C_3\|f\|}{M} \right]\varepsilon := N_1\|x-y\| + M_3\varepsilon,$$

In the same way, it gives

$$|D_{0^+}^{\alpha}x(t) - D_{0^+}^{\alpha}y(t)| \leq \int_0^t \left| \frac{(t-s)^{\eta-\alpha-1}}{\Gamma(\eta-\alpha)} f(s,x(s),D_{0^+}^{\alpha}x(s)) - \frac{(t-s)^{\widetilde{\eta}-\alpha-1}}{\Gamma(\widetilde{\eta}-\alpha)} f(s,y(s),D_{0^+}^{\alpha}y(s)) \right| \mathrm{d}s$$
$$+ \frac{1}{M}\int_0^1 \left| \frac{\Gamma(\eta)\Gamma(\eta+\beta)t^{\eta-\alpha-1}(1-s)^{\eta-\alpha-1}}{\Gamma(\eta-\alpha)} f(s,x(s),D_{0^+}^{\alpha}x(s)) - \frac{\Gamma(\widetilde{\eta})\Gamma(\widetilde{\eta}+\beta)t^{\widetilde{\eta}-\alpha-1}(1-s)^{\widetilde{\eta}-\alpha-1}}{\Gamma(\widetilde{\eta}-\alpha)} f(s,y(s),D_{0^+}^{\alpha}y(s)) \right| \mathrm{d}s$$
$$+ \frac{\tau}{M}\int_0^{\xi} \Big| \Gamma(\eta)t^{\eta-\alpha-1}(\xi-s)^{\eta+\beta-1} f(s,x(s),D_{0^+}^{\alpha}x(s)) - \Gamma(\widetilde{\eta})t^{\widetilde{\eta}-\alpha-1}(\xi-s)^{\widetilde{\eta}+\beta-1} f(s,y(s),D_{0^+}^{\alpha}y(s)) \Big| \mathrm{d}s$$
$$\leq \left[\frac{1}{\Gamma(\eta-\alpha+1)} + \frac{\Gamma(\eta)\Gamma(\eta+\beta)}{M\Gamma(\eta-\alpha)} + \frac{\tau\Gamma(\eta)}{M} \right](L_3+L_4)\|x-y\| + \left[C_4\|f\| + \frac{C_5\|f\|}{M} + \frac{\tau C_6\|f\|}{M} \right]\varepsilon := N_2\|x-y\| + M_4\varepsilon,$$

where

$$C_4 = \max_{x\in[\eta,\widetilde{\eta}]} \left\{ \frac{t^{x-\alpha-1}}{\Gamma(x-\alpha)}, \ 0<t<1 \right\},$$

$$C_5 = \max_{x\in[\eta,\widetilde{\eta}]} \left\{ \left(\frac{\Gamma(x)\Gamma(x+\beta)(t-ts)^{x-\alpha-1}}{\Gamma(x-\alpha)} \right)', \ 0<t, \ s<1 \right\},$$

$$C_6 = \max_{x\in[\eta,\widetilde{\eta}]} \left\{ \left(\Gamma(x)t^{x-\alpha-1}(\xi-s)^{x+\beta-1} \right)', \ 0<t, \ \xi, s<1, \ s<\xi \right\},$$

$$M_4 = C_4\|f\| + \frac{C_5\|f\|}{M} + \frac{\tau C_6\|f\|}{M}.$$

Thus

$$\|x(t)-y(t)\| \leq \frac{M_3+M_4}{1-(N_1+N_2)}\varepsilon.$$

When the integral order of boundary value condition β changes slightly, the stability of the solution is as follows. □

Theorem 5. Let x be the solution of equation (1), and let y be the solution of the following problem:

$$\begin{cases} D_{0^+}^{\eta}y(t) + f(t,y(t),D_{0^+}^{\alpha}y(t)) = 0, \\ y^{(i)}(0) = 0, \quad i = 0,1,\cdots,n-2, \\ D_{0^+}^{\alpha}y(1) = \widetilde{\tau}I_{0^+}^{\beta}y(\xi) = \widetilde{\tau}\displaystyle\int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} y(s)\,\mathrm{d}s, \end{cases}$$

where $|\tau-\widetilde{\tau}|<\varepsilon$, ε is an any small constant. Then,

$$\|x-y\| \leq \frac{[\Gamma(\eta)+\Gamma(\eta-\alpha)]\|f\|}{M[1-(N_1+N_2)]}\varepsilon.$$

Proof. For any $t \in [0,1]$, we have

$$|x(t)-y(t)| \le \Big[\frac{1}{\Gamma(\eta+1)}+\frac{\Gamma(\eta+\beta)}{M}+\frac{\tau\Gamma(\eta-\alpha)}{M}\Big](L_1+L_2) \cdot \|x-y\|+\frac{\Gamma(\eta-\alpha)\|f\|}{M}|\tau-\widetilde{\tau}| := N_1\|x-y\|+\frac{\Gamma(\eta-\alpha)\|f\|}{M}|\tau-\widetilde{\tau}|,$$

$$|D^{\alpha}_{0^+}x(t)-D^{\alpha}_{0^+}y(t)| \le \Big[\frac{1}{\Gamma(\eta-\alpha+1)}+\frac{\Gamma(\eta)\Gamma(\eta+\beta)}{M\Gamma(\eta-\alpha)} + \frac{\tau\Gamma(\eta)}{M}\Big](L_1+L_2)\|x-y\| + \frac{\Gamma(\eta)\|f\|}{M}|\tau-\widetilde{\tau}| := N_2\|x-y\|+\frac{\Gamma(\eta)\|f\|}{M}|\tau-\widetilde{\tau}|.$$

Thus, $\|x(t)-y(t)\| \le \dfrac{[\Gamma(\eta)+\Gamma(\eta-\alpha)]\|f\|}{M[1-(N_1+N_2)]}\varepsilon$.

When the boundary parameter ξ changes slightly, the stability of the solution is as follows. □

Theorem 6. Let x be the solution of equation (1), and let y be the solution of the following problem:

$$\begin{cases} D^{\eta}_{0^+}y(t)+f(t,y(t),D^{\alpha}_{0^+}y(t))=0, \\ y^{(i)}(0)=0, \quad i=0,1,\cdots,n-2, \\ D^{\alpha}_{0^+}y(1)=\tau I^{\widetilde{\beta}}_{0^+}y(\xi)=\tau\displaystyle\int_0^{\xi}\frac{(\xi-s)^{\widetilde{\beta}-1}}{\Gamma(\widetilde{\beta})}y(s)\,\mathrm{d}s, \end{cases}$$

where $|\beta-\widetilde{\beta}|<\varepsilon$, ε is an any small constant. Then, it holds that

$$\|x-y\| \le \frac{[\Gamma(\eta)+\Gamma(\eta-\alpha)]C_7\|f\|+\tau\Gamma(\eta-\alpha)\Gamma(\eta)C_8\|f\|}{M\Gamma(\eta-\alpha)[1-(N_1+N_2)]}\varepsilon.$$

Proof. For any $t\in[0,1]$, it gives

$$|x(t)-y(t)| \le \Big[\frac{1}{\Gamma(\eta+1)}+\frac{\Gamma(\eta+\beta)}{M}+\frac{\tau\Gamma(\eta-\alpha)}{M}\Big](L_1+L_2) \cdot\|x-y\|+\frac{C_7\|f\|}{M}\varepsilon := N_1\|x-y\|+\frac{C_7\|f\|}{M}\varepsilon,$$

where $C_7=\max\limits_{x\in[\beta,\widetilde{\beta}]}\Big\{(\Gamma(\eta+x))'\Big\}$. What is more, one has that

$$|D^{\alpha}_{0^+}x(t)-D^{\alpha}_{0^+}y(t)| \le \Big[\frac{1}{\Gamma(\eta-\alpha+1)}+\frac{\Gamma(\eta)\Gamma(\eta+\beta)}{M\Gamma(\eta-\alpha)} + \frac{\tau\Gamma(\eta)}{M}\Big](L_1+L_2)\|x-y\| + \frac{\Gamma(\eta)C_7\|f\|}{M\Gamma(\eta-\alpha)}\varepsilon+\frac{\tau\Gamma(\eta)C_8\|f\|}{M}\varepsilon := N_2\|x-y\|+\frac{\Gamma(\eta)C_7\|f\|}{M\Gamma(\eta-\alpha)}\varepsilon+\frac{\tau\Gamma(\eta)C_8\|f\|}{M}\varepsilon,$$

where $C_8=\max\limits_{x\in[\beta,\widetilde{\beta}]}\left\{\dfrac{\xi^{\eta+x-2}}{\eta+x-1}\right\}$. Thus, we shall get

$$\|x(t)-y(t)\| \le \Big(\frac{[\Gamma(\eta)+\Gamma(\eta-\alpha)]C_7\|f\|}{M\Gamma(\eta-\alpha)} + \frac{\tau\Gamma(\eta-\alpha)\Gamma(\eta)C_8\|f\|}{M\Gamma(\eta-\alpha)}\Big)[1-(N_1+N_2)]\varepsilon.$$

When the boundary value τ changes slightly, the stability of the solution is as follows. □

Theorem 7. Let x be the solution of equation (1), and let y be the solution of the following problem:

$$\begin{cases} D^{\eta}_{0^+}y(t)+f(t,y(t),D^{\alpha}_{0^+}y(t))=0, \\ y^{(i)}(0)=0, \quad i=0,1,\cdots,n-2, \\ D^{\alpha}_{0^+}y(1)=\tau I^{\beta}_{0^+}y(\widetilde{\xi})=\tau\displaystyle\int_0^{\widetilde{\xi}}\frac{(\widetilde{\xi}-s)^{\beta-1}}{\Gamma(\beta)}y(s)\,\mathrm{d}s, \end{cases}$$

where $|\xi-\widetilde{\xi}|<\varepsilon$, ε is an any small constant. Then,

$$\|x-y\| \le \frac{\tau\Gamma(\eta)\|f\|[C_9+(\eta+\beta-1)]}{M(\eta+\beta-1)[1-(N_1+N_2)]}\varepsilon.$$

Proof. For any $t\in[0,1]$, one gets

$$|x(t)-y(t)| \le \Big[\frac{1}{\Gamma(\eta+1)}+\frac{\Gamma(\eta+\beta)}{M}+\frac{\tau\Gamma(\eta-\alpha)}{M}\Big](L_1+L_2)\|x-y\|+\frac{\tau\Gamma(\eta-\alpha)C_9\|f\|}{M(\eta+\beta-1)}\varepsilon+\frac{\tau\Gamma(\eta-\alpha)\|f\|}{M}\varepsilon := N_1\|x-y\|+\frac{\tau\Gamma(\eta-\alpha)\|f\|[C_9+(\eta+\beta-1)]}{M(\eta+\beta-1)}\varepsilon,$$

where $C_9=\max\limits_{x\in[\xi,\widetilde{\xi}]}\Big\{(x-s)^{\eta+\beta-2}\Big\}$. Moreover, we conclude

$$|D^{\alpha}_{0^+}x(t)-D^{\alpha}_{0^+}y(t)| \le \Big[\frac{1}{\Gamma(\eta-\alpha+1)}+\frac{\Gamma(\eta)\Gamma(\eta+\beta)}{M\Gamma(\eta-\alpha)} + \frac{\tau\Gamma(\eta)}{M}\Big](L_1+L_2)\|x-y\| + \frac{\tau\Gamma(\eta)C_9\|f\|}{M(\eta+\beta-1)}\varepsilon+\frac{\tau\Gamma(\eta)\|f\|}{M}\varepsilon := N_2\|x-y\|+\frac{\tau\Gamma(\eta)\|f\|[C_9+(\eta+\beta-1)]}{M(\eta+\beta-1)}\varepsilon.$$

Thus, one observes

$$\|x(t)-y(t)\| \le \frac{\tau\Gamma(\eta)\|f\|[C_9+(\eta+\beta-1)]}{M(\eta+\beta-1)[1-(N_1+N_2)]}\varepsilon.$$

According to Theorem 4–Theorem 7, we can see that the change of the solution when the main parameters of the equation are slightly perturbed, that is, the solution depends on the main parameters in a continuous way. □

5. APPLICATIONS

Example 1. Consider the following equation:

$$\begin{cases} D_{0^+}^{\frac{13}{4}}x(t)+f\left(t,x(t),D_{0^+}^{\frac{4}{3}}x(t)\right)=0, \\ x(0)=0,\ \ x'(0)=0,\ \ x''(0)=0, \\ D_{0^+}^{\frac{4}{3}}x(1)=\frac{1}{6}I_{0^+}^{\frac{4}{5}}x\left(\frac{1}{10}\right)=\frac{1}{6}\int\limits_0^{\frac{1}{10}}\frac{\left(\frac{1}{10}-s\right)^{-\frac{1}{5}}}{\Gamma\left(\frac{4}{5}\right)}x(s)\,\mathrm{d}s, \end{cases} \tag{6}$$

here $\eta=\frac{13}{4}\in(3,4)$, $\alpha=\frac{4}{3}\in(1,2)$, $\beta=\frac{4}{5}\in(0,1)$, $\tau=\frac{1}{6}$, $\xi=\frac{1}{10}$, $f(t,x,y)=\frac{1}{2}t^2+\frac{92t}{103\pi}x+\frac{1}{100}y$.

Proof

Conclusion 1. Equation (6) has at least one solution.

(a) From the above parameters, we can see that

$$M=\Gamma\left(\frac{13}{4}\right)\Gamma\left(\frac{81}{20}\right)-\frac{1}{6}\Gamma\left(\frac{13}{4}\right)\Gamma\left(\frac{23}{12}\right)\left(\frac{1}{10}\right)^{\frac{61}{20}}=16.2924,$$

$$M_1=\frac{\Gamma(\eta+\beta)}{M}=\frac{\Gamma\left(\frac{81}{20}\right)}{M}\approx 0.3923,$$

$$M_2=\frac{\Gamma(\eta)\Gamma(\eta+\beta)}{M\Gamma(\eta-\alpha)}=\frac{\Gamma\left(\frac{13}{4}\right)\Gamma\left(\frac{81}{20}\right)}{M\Gamma\left(\frac{23}{12}\right)}\approx 1.0335.$$

(b) From the definition of f, we can deduce

$$\begin{aligned}|f(t,x,y)|&\le\frac{t}{2}+\frac{92t}{103\pi}|x|+\frac{1}{100}|y|\\&=L_0(t)+L_1(t)|x|+L_2(t)|y|,\end{aligned}$$

$$M_1\int\limits_0^1(L_1(t)+L_2(t))\,\mathrm{d}t=\frac{\Gamma\left(\frac{81}{20}\right)}{M}\left(\frac{92}{206\pi}+\frac{1}{100}\right)<\frac{1}{2},$$

$$M_2\int\limits_0^1(L_1(t)+L_2(t))\,\mathrm{d}t=\frac{\Gamma\left(\frac{13}{4}\right)\Gamma\left(\frac{81}{20}\right)}{M\Gamma\left(\frac{23}{12}\right)}\left(\frac{92}{206\pi}+\frac{1}{100}\right)<\frac{1}{2}.$$

According to Theorem 1, equation (6) has at least one solution.

Conclusion 2. Equation (6) is Hyers-Ulam stable.

From the definition of f, we have

$$\begin{aligned}|f(t,x,y)-f(t,\bar{x},\bar{y})|&\le\frac{92t}{103\pi}|x-\bar{x}|+\frac{1}{100}|y-\bar{y}|\\&=L_3|x-\bar{x}|+L_4|y-\bar{y}|.\end{aligned} \tag{7}$$

According to Theorem 3, equation (6) is Hyers-Ulam stable.

Conclusion 3. Equation (6) has a unique solution.

From (7) and

$$\max\{M_1,M_2\}\cdot\int\limits_0^1(L_3(t)+L_4(t))\,\mathrm{d}t=\frac{\Gamma\left(\frac{13}{4}\right)}{M}\frac{\Gamma\left(\frac{81}{20}\right)}{\Gamma\left(\frac{23}{12}\right)}\cdot\left(\frac{92}{206\pi}+\frac{1}{100}\right)<1,$$

then by Theorem 2, equation (6) has a unique solution.

Conclusion 4. The figure simulation of unique solution to equation (6) is given.

From Lemma 3, we can know that the solution of equation (6) has following form

$$x(t)=\int\limits_0^1 G(t,s)f(s,x(s),D_{0^+}^{\frac{4}{3}}x(s))\,\mathrm{d}s.$$

Let $x_0=t^{\frac{9}{4}}$ and an iterative schemes

$$x_n(t)=\int\limits_0^1 G(t,s)f(s,x_{n-1}(s),D_{0^+}^{\frac{4}{3}}x_{n-1}(s))\,\mathrm{d}s, \tag{8}$$

be a basis numerical algorithms, where $n=1,2,\ldots$

Due to the large amount of calculation, we only show the results of three iterations in this article. Although the number of iterations is small, it can be seen from the figure that the error between the second iteration result and the third iteration result is relatively small. To some extent, the third iteration result can show the properties of the unique solution for equation (6). In addition, the result also proves the effectiveness of the iterative scheme from the side. The figure simulation of 1st iteration result x_1, 2nd iteration result x_2, and 3rd iteration result x_3 is shown in Fig. 1.

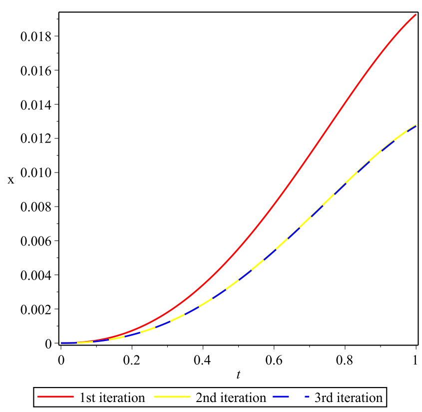

Fig. 1. Numerical solution of equation (6) by iterative formula $x_{n+1}=\int\limits_0^1 G(t,s)f(s,x_n(s),x_n(s))\,\mathrm{d}s$, the initial function was set as $x_0=t^{\frac{9}{4}}$

Conclusion 5. When there are some small perturbations in the parameters of equation (6), the variation diagram of unique solution is given.

Using the above method, we get three iterative results, when η has disturbances, i.e. $\eta-\frac{1}{10}$, $\eta-\frac{1}{50}$, η, $\eta+\frac{1}{50}$, and $\eta+\frac{1}{10}$, which are shown in Fig. 2.

Similarly, we give the image of the solution when the remaining parameters change. We find that small changes in parameters β, ξ, τ have little impact on the results of the three iterations, therefore we only give the trend diagram of the first iteration, which are shown in Fig. 3. Since the change trend of the unique solution are not obvious when β, ξ, τ change, we give the local graph to show more clearly the difference, which are shown in Fig. 4. Clearly, by Fig. 2, we can see that as η increases, the value of x_1, x_2, and x_n decrease, i.e. η has a negative correlation with the unique solution of equation (6). From the (a) of Fig. 3, as β increases, the value of x_1 decrease, i.e. β has a negative correlation with the unique solution. From the (b) of Fig. 3 and (a) of Fig. 4, as ξ increases, the value of x_1 decrease, i.e. ξ has a negative correlation with the unique solution. From the (c) of Fig. 3 and (b) of Fig. 4, as τ increases, the value of x_1 decrease, i.e. τ has a negative correlation with the unique solution.

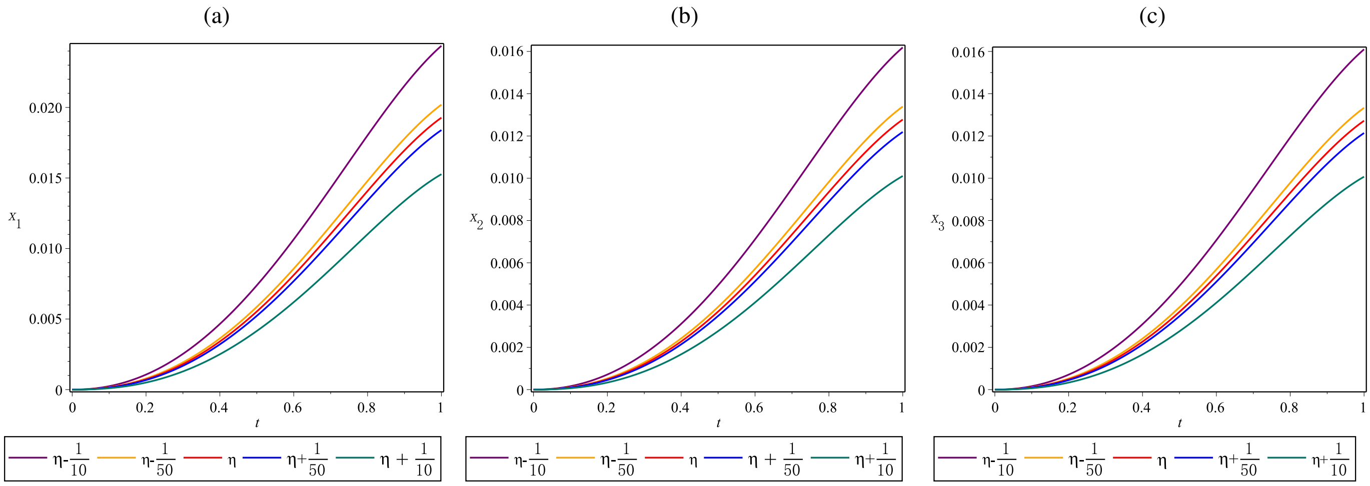

Fig. 2. Numerical iterative solution x_n of equation (6) when $\eta-\frac{1}{10}$, $\eta-\frac{1}{50}$, $\eta=\frac{13}{4}$, $\eta+\frac{1}{50}$, and $\eta+\frac{1}{10}$. (a) $n=1$, (b) $n=2$, and (c) $n=3$

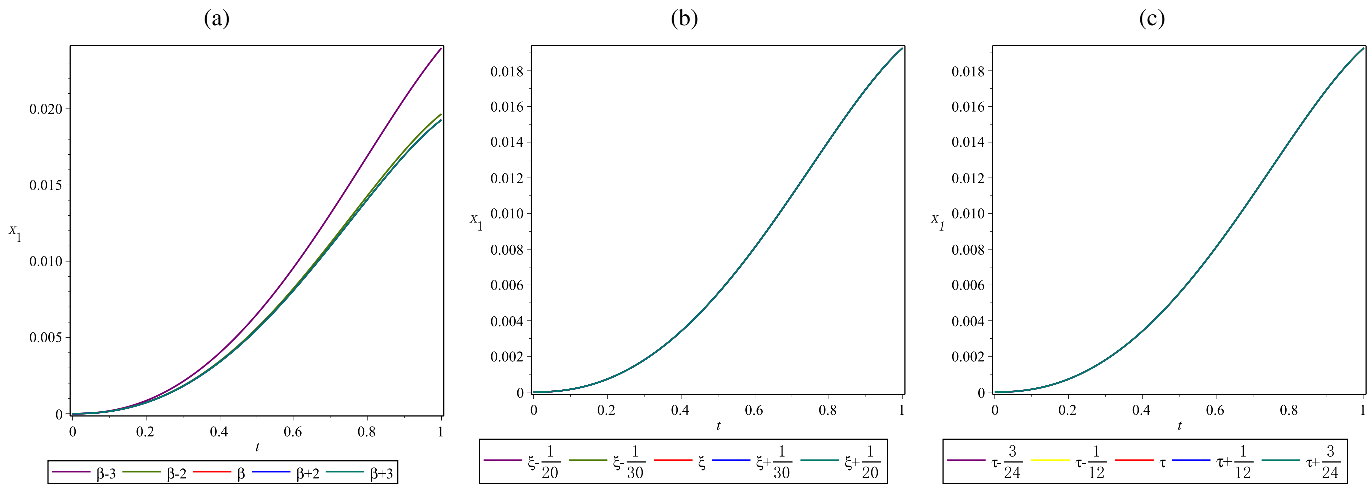

Fig. 3. 1st iteration result x_1 when there is a small change in β, ξ, τ.
(a) $\beta-\frac{3}{5}$, $\beta-\frac{3}{10}$, $\beta=\frac{4}{5}$, $\beta+\frac{3}{10}$, and $\beta+\frac{3}{5}$,
(b) $\xi-\frac{1}{20}$, $\xi-\frac{1}{30}$, $\xi=\frac{1}{10}$, $\xi+\frac{1}{20}$, and $\xi+\frac{1}{30}$,
(c) $\tau-\frac{3}{24}$, $\tau-\frac{1}{12}$, $\tau=\frac{1}{6}$, $\tau+\frac{1}{12}$, and $\tau+\frac{3}{24}$

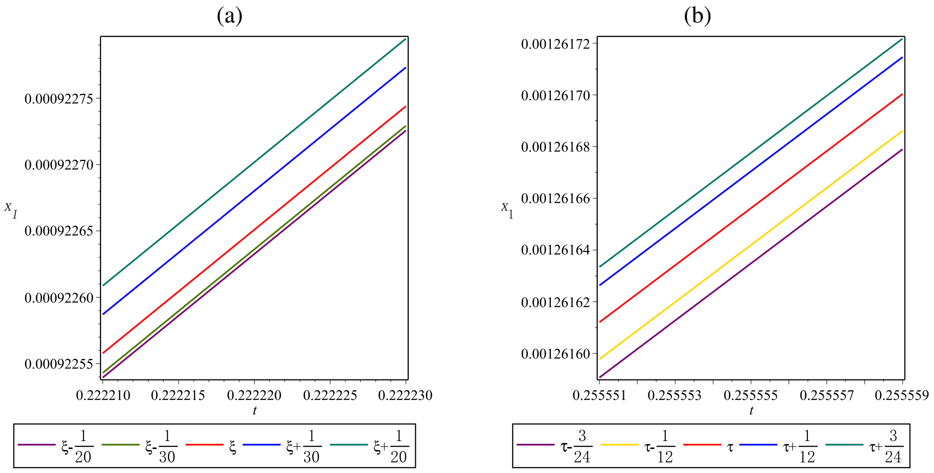

Fig. 4. Local graph od 1st iteration result x_1 when there is a small change in ξ, τ. (a) $\xi-\frac{1}{20}$, $\xi-\frac{1}{30}$, $\xi=\frac{1}{10}$, $\xi+\frac{1}{20}$, and $\xi+\frac{1}{30}$, (b) $\tau-\frac{3}{24}$, $\tau-\frac{1}{12}$, $\tau=\frac{1}{6}$, $\tau+\frac{1}{12}$, and $\tau+\frac{3}{24}$

ACKNOWLEDGEMENTS

The authors declare that they have no competing interests. Authors would like to appreciate support from Human Sciences Research Council (HSRC) for giving us the platform to collaborate for this work. The first and second author acknowledge the support of Key R&D program of Shanxi Province (International Cooperation, 201903D421042), The Research Project Supported by Shanxi Scholarship Council of China (2021-030), and Shanxi Youth Scientific Research Project (202103021223060).

REFERENCES

[1] K. Diethelm and N.J. Ford, "Analysis of fractional differential equations," *J. Math. Anal. Appl.*, vol. 265, no. 2, pp. 229–248, 2002.

[2] C.B. Zhai and W. Wang, "Properties of positive solutions for m-point fractional differential equations on an infinite interval," *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, vol. 113, no. 2, pp. 1289–1298, 2019.

[3] T. Kaczorek, J. Klamka, and A. Dzieliński, "Controllability of linear convex combination of linear discrete-time fractional systems," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 70, p. e143102, 2022.

[4] M. Klimek, "Existence-uniqueness result for a certain equation of motion in fractional mechanics," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 58, no. 4, pp. 573–581, 2010.

[5] M.A. Zaky, A.S. Hendy, and D. Suragan, "A note on a class of Caputo fractional differential equations with respect to another function," *Math. Comput. Simul.*, vol. 196, pp. 289–295, 2022.

[6] R. Magin, "Fractional calculus in bioengineering, part 1," in *Crit. Rev. Biomed. Eng.*, vol. 32, no. 1, 2004.

[7] H.G. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Chen, "A new collection of real world applications of fractional calculus in science and engineering," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 64, pp. 213–231, 2018.

[8] I. Slimane, G. Nazir, J.J. Nieto, and F. Yaqoob, "Mathematical analysis of Hepatitis C Virus infection model in the framework of non-local and non-singular kernel fractional derivative," *Int. J. Biomath.*, vol. 16, p. 2250064, 2023.

[9] S. Abbas, M. Benchohra, J.E. Lazreg, J.J. Nieto, and Y. Zhou, *Fractional differential equations and inclusions*, Classical and Advanced Topics. World Scientific, 2023.

[10] H. Boulares, A. Ardjouni, and Y. Laskri, "Positive solutions for nonlinear fractional differential equations," *Positivity*, vol. 21, pp. 1201–1212, 2017.

[11] B. Ahmad and R. Luca, "Existence of solutions for a system of fractional differential equations with coupled nonlocal boundary conditions," *Fract. Calc. Appl. Anal.*, vol. 21, no. 2, pp. 423–441, 2018.

[12] S. Belmor, C. Ravichandran, and F. Jarad, "Nonlinear generalized fractional differential equations with generalized fractional integral conditions," *J. Taibah Univ. Sci.*, vol. 14, no. 1, pp. 114–123, 2020.

[13] H.M. Srivastava, A. Shehata, and S.I. Moustafa, "Some fixed point theorems for-contractions and their application to fractional differential equations," *Russ. J. Math. Phys.*, vol. 27, no. 3, pp. 385–398, 2020.

[14] H. Boulares, A. Benchaabane, N. Pakkaranang, R. Shafqat, and B. Panyanak, "Qualitative properties of positive solutions of a kind for fractional pantograph problems using technique fixed point theory," *Fractal Fract.*, vol. 6, no. 10, p. 593, 2022.

[15] M.A. Almalahi, S.K. Panchal, and F. Jarad, "Stability results of positive solutions for a system of ψ-Hilfer fractional differential equations," *Chaos Solitons Fractals*, vol. 147, p. 110931, 2021.

[16] M. Xu and S. Sun, "Positivity for integral boundary value problems of fractional differential equations with two nonlinear terms," *J. Appl. Math. Comput.*, vol. 59, pp. 271–283, 2019.

[17] J.R. Wang, Y. Zhou, and W. Wei, "Study in fractional differential equations by means of topological degree methods," *Numer. Funct. Anal. Optim.*, vol. 33, no. 2, pp. 216–238, 2012.

[18] I. Ullah, R. Khan, and K. Shah, "On using topological degree theory to investigate a coupled system of non linear hybrid differential equations," *Comput. Methods Differ. Equ.*, vol. 7, no. 2, pp. 224–234, 2019.

[19] G.S. Ladde, V. Lakshmikantham, and A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston, 1985

[20] J.J. Nieto, "An abstract monotone iterative technique", *Nonlinear Anal.-Theory Methods Appl.*, vol. 28, no. 12, pp. 1923–1933, 1997.

[21] Z. Baitiche, C. Derbazi, M. Benchohra, and J.J. Nieto, "Monotone iterative technique for a new class of nonlinear sequential fractional differential equations with nonlinear boundary conditions under the ψ-Caputo operator," *Mathematics*, vol. 10, no. 7, p. 1173, 2022.

[22] A. Ben Makhlouf, D. Boucenna, and M.A. Hammami, "Existence and stability results for generalized fractional differential equations," *Acta Math. Sci.*, vol. 40, pp. 141–154, 2020.

[23] A. Khan, H. Khan, J.F. Gómez-Aguilar, and T. Abdeljawad, "Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel," *Chaos Solitons Fractals*, vol. 127, pp. 422–427, 2019.

[24] A. Cardone and D. Conte, "Stability analysis of spline collocation methods for fractional differential equations," *Math. Comput. Simul.*, vol. 178, pp. 501–514, 2020.

[25] X. Wu, F. Chen, and S. Deng, "Hyers-Ulam stability and existence of solutions for weighted Caputo-Fabrizio fractional differential equations," *Chaos Solitons Fractals-X* , vol. 5, p. 100040, 2020.

[26] J.V.C. Sousa, F.G. Rodrigues, and E. Capelas de Oliveira, "Capelas de Oliveira. Stability of the fractional Volterra integro-differential equation by means of ψ-Hilfer operator," *Math. Meth. Appl. Sci.*, vol. 42, no. 9, pp. 3033–3043, 2019.

[27] R. Agarwal, D. O'Regan, and S. Hristova, "Stability of Caputo fractional differential equations by Lyapunov functions," *Appl. Math.*, vol. 60, no. 6, pp. 653–676, 2015.

[28] K. Liu, J.R. Wang, Y. Zhou, and D. O'Regan, "Hyers–Ulam stability and existence of solutions for fractional differential equations with Mittag–Leffler kernel," *Chaos Solitons Fractals*, vol. 132, p. 109534, 2020.