Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(6): 10174–10199.
DOI: 10.3934/mbe.2023446
Received: 11 January 2023
Revised: 19 March 2023
Accepted: 21 March 2023
Published: 29 March 2023

Research article

Modelling the impact of vaccination and environmental transmission on the dynamics of monkeypox virus under Caputo operator

Emmanuel Addai[1,2], **Mercy Ngungu**[3,*], **Musibau Abayomi Omoloye**[4] **and Edmore Marinda**[3]

[1] College of Biomedical Engineering, Taiyuan University of Technology, Shanxi Taiyuan 030024, China
[2] Department of Mathematics, Taiyuan University of Technology, Shanxi Taiyuan 030024, China
[3] Human Sciences Research Council (HSRC), South Africa
[4] Department of Mathematics and Computer Science, Elizade University, Ilara-Mokin, Ondo State, Nigeria

* **Correspondence:** Email: MNgungu@hsrc.ac.za.

Abstract: In this study, we examine the impact of vaccination and environmental transmission on the dynamics of the monkeypox. We formulate and analyze a mathematical model for the dynamics of monkeypox virus transmission under Caputo fractional order. We obtain the basic reproduction number, the conditions for the local and global asymptotic stability for the disease-free equilibrium of the model. Under the Caputo fractional order, existence and uniqueness solutions have been determined using fixed point theorem. Numerical trajectories are obtained. Furthermore, we explored some of the sensitive parameters impact. Based on the trajectories, we hypothesised that the memory index or fractional order could use to control the Monkeypox virus transmission dynamics. We observed that if the proper vaccination is administrated, public health education is given, and practice like personal hygiene and proper disinfection spray, the infected individuals decreases.

Keywords: Caputo fractional derivative; reproduction number; global asymptotic stability; numerical scheme

1. Introduction

Over 100 nonendemic nations globally reported an unprecedented and unexpected increase in the number of human monkeypox cases in 2022 [1]. The monkeypox virus (MPXV), which is responsible for this zoonotic disease also known as monkeypox (MPOX), was formerly endemic in sub-Saharan Africa countries [1–3]. Prior to the present of 2022 MPOX outbreak, MPXV was only mildly spreading, mostly in endemic nations with secondary attack rates that only ever went above 10% [3–5]. With over 80,000 cases reported in more than 100 formerly nonendemic countries in 2022, the dynamics of MPOX spread have significantly changed [2, 6]. The World Health Organization (WHO) was compelled by this circumstance to proclaim the 2022 MPOX outbreak a public health emergency of worldwide concern on July 23, 2022 [7].

The first MPXV case in humans was reported in sub-Saharan Africa country, the Democratic Republic of Congo in 1970 [2,3,6], and the MPXV was initially identified at an animal facility in Denmark in 1958 [2,3,6]. The majority of cases of MPOX have only been reported in sub-Saharan Africa. In 2003, the disease spread outside of the sub-Saharan Africa. The lack of smallpox vaccination programmes after variola eradication presumably reduced immunity, there were more encounters between people and wildlife, and MPXV may have changed to allow for sustained human-to-human transmission [7–11]. One of the main modes of transmission for the MPXV is the environmental viral load. Direct contact with items or materials contaminated with MPXV contributes considerably to the transmission of the MPOX. Virus particles can survive for many months if contaminated material is kept in a low-humidity, low-temperature environment that is shielded from UV light [12, 13]. Some people who may have contracted MPOX have fever, headache, muscle aches, backaches, swollen lymph nodes, chills, and fatigue. The bulk of deaths from MPOX occur in children under the age of ten, accounting for up to 10% of those who contract the disease [9]. The need for suitable response measures is mandated by the evolving epidemiology of MPOX [10]. These actions include giving vaccinations to high-risk groups and engaging in educational initiatives to raise MPOX awareness and knowledge [2, 11].

The WHO emphasized the significance of strong public health control measures to stop the spread of MPXV [14], despite the fact that there is no specific vaccine or medication for the management and prevention of MPOX [9]. This entails improving the ability of healthcare professionals to recognize cases by implementing early diagnosis and delivering organized, efficient medical care [14]. One requirement for limiting this outbreak is the general public to comprehend MPOX. Prior studies have shown that the populace of Saudi Arabia, Indonesia, and Jordan have a generally low level of knowledge about MPOX [15–17]. Nearly one-quarter of dental practitioners in a recent poll in a northern Indian state said they had never heard of MPOX [18]. Concerns were expressed about the population's level of MPOX immunisation during the epidemic. Therefore, it is essential to ascertain the population's readiness to receive vaccinations and willingness to pay for them before widespread immunisation. A poll carried out in Indonesia revealed that 96% of the frontline doctors were open to receiving a free MPOX vaccination [19]. Currently, a survey [20] for the Chinese population's knowledge of MPOX and vaccination intention revealed that more knowledge of MPOX was associated with greater willingness to take vaccine. People who suffered from chronic diseases and had high salaries were more willing to pay more for immunisations.

Fractional order modeling is a powerful method that has been used to study the nature of diseases transmissions. Recently, some authors have used fractional models to study infectious and non-infectious diseases transmission dynamics. This is because the dynamic transmissions that take place in biological models can be accurately modeled using fractional calculus. Additionally, the fractional order derivation has a global dimension as opposed to the local identity shared by the integer order derivation. The most efficient and useful techniques for representing the nonlinear processes that show up in countless applications to real-world settings are now fractional derivative order differential problems. The fractional calculus has improved the modelling precision of many phenomena in the physical sciences. Primarily, Caputo fractional derivative, the Caputo-Fabrizio (CF) derivative, and Atangana-

Baleanu-Caputo (ABC) have lately been used in the field of mathematical biology. The only reason that Caputo, CF, and ABC differed from one another in terms of their fractional derivatives was that Caputo is defined by a power law, CF is defined by utilising an exponential decay rule, and ABC is defined by the Mittag-Leffler law [21–25]. These operators have numerous works recently, for instance, in [26], the Mittag-Leffler type kernel modeling for Ebola-malaria co-infection was investigated. In [27], Singular and non-singular fractional operators was used to explore the COVID-19 model. In [28], mathematical model for HIV/AIDS using the Mittag-Leffler type kernel was investigated. Furthermore, in the context of MPOX transmission dynamics, the authors in [39] used real data from Nigeria to study the dynamics of the transmission of MPOX virus using fractional calculus. They studied the infection control policies which will help the general public comprehension of the importance of control parameters in the extinction of the disease. Also, in [30] and [31] studied the transmission dynamics of the MPOX virus with non-pharmaceutical intervention. The Caputo derivative, as opposed to the Riemann Liouville fractional derivative, permits the use of conventional beginning and steady state in the derivation and has the advantage that the derivative of a constant is 0. Due to this advantage many researchers in the field of mathematical biology take into consideration. The authors in [32] modelled giving up smoking mathematically under Caputo fractional derivative. In [33], the authors investigated Caputo fractional model for Middle East Lungs Coronavirus dynamism transmission. For more papers on Caputo fractional derivative see, for instance, [34, 35] and references therein.

This study aims to investigate the public health approach based on vaccination campaigns and environmental transmission controls in the successful control of the MPXV. For the purpose of this study, we assume vaccinated vaccine as JYNNEOS, a 2-dose vaccine with 79% efficacy. Although some mathematical models have been developed to evaluate the effectiveness of isolation and quarantine-based anti-MPXV control techniques. The aforementioned articles made it clear that no mathematical model for the MPXV considered how vaccination and environmental transmission impact the disease spread. In light of this, the current study creates a mathematical model for the dynamics of MPXV transmission that improved upon earlier studies in the following areas:

i. A novel Caputo fractional epidemiological model that takes vaccination and environmental transmission into account is taken into consideration and studied.
ii. Employing the fixed point theorem, the existence and uniqueness of the solution of the MPXV transmission dynamics incorporating vaccination and environmental transmission has been achieved.
iii. When recollection of the prior history of the MPXV transmission is taken into consideration through simulations, we showed the rich transmission dynamics of the MPXV using the Caputo fractional order derivative.
v. We emphasize the significance of vaccine rate and efficacy, rates of viral contribution from human and rodent to environment, and the rate at which humans are exposed to certain classes of infection on the MPXV transmission dynamics when the fractional order is 0.95.

The rest of this study is structured as follows: Section 2 presents briefly some key ideas, fundamental definitions, and preliminary findings. We formulate the suggested model both fractionally and non-fractionally in Section 3 and briefly describe each parameters. The analysis of fundamental mathematical models, including positivity, invariance, boundedness, disease-free equilibrium (DFE), reproduction number, local and global stabilities, are covered in Section 4. In Section 5, we look at the dynamics of vaccination and environmental transmission as they relate to the existence and uniqueness of the MPXV. Sections 6 and 7 deal with the numerical framework and simulations, respectively. Section 8 of the paper conclude the work.

2. Preliminaries

In this section, we review several key definitions, lemmas, and concepts that are necessary to understand our suggested model.

Definition 2.1 [34, 35]. Given a function $u : \mathbb{R}^+ \to \mathbb{R}$, and $\alpha \in (n-1, n), n \in N$. The left Caputo fractional derivative of order α of the function u is defined as

$$ {}_0^C D_t^{\alpha}(u(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t u^n(\Theta)(t-\Theta)^{n-\alpha-1} d\Theta $$

Definition 2.2 [35]. The corresponding Riemann-Liouville fractional integral associated with the power-law kernel is defined as

$$ {}_0^C I_t^{\alpha}(u(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\Theta)^{\alpha-1} u(\Theta) d\Theta, t > 0. $$

Lemma 2.3 [36]. Assuming there is a function $u(t) \in C[0, \eta]$ of order $\alpha \in (0, 1)$, then the solution of fractional differential equation

$$ \begin{cases} {}_0^C D_t^{\alpha} u(t) = \Upsilon(t, u(t)), t \in [0, \eta], \\ u(0) = u_0, \end{cases} $$

is given by

$$ u(t) - u(0) = \frac{1}{\Gamma(\alpha)} \int_0^t \Upsilon(\Theta, u(\Theta))(t-\Theta)^{\alpha-1} d\Theta. $$

3. Model Formulation

We investigate the populations of humans and rodents in a closed homogeneous habitat using a system of differential equations. A population of humans has four compartments of total size $N_h(t)$: Susceptible $S_h(t)$, Exposed $E_h(t)$; Vaccinated $V_h(t)$; Infected $I_h(t)$; Recovered $R_h(t)$; where $N_h(t) = S_h(t) + V_h(t) + E_h(t) + I_h(t) + R_h(t)$. And rodent population $N_r(t)$ is split into $S_r(t)$ Susceptible ; $E_r(t)$ Expose ; $I_r(t)$ Infected ; Let $N_r(t) = S_r(t) + E_r(t) + I_r(t)$. We introduce another compartment $L(t)$ that represents the environmental factors that contributes to the emergence of the virus. The hypothesized dynamics of virus transmission are described by the ordinary differential equations in model (3.1) based on the foregoing description;

$$\begin{cases} \frac{dS_h}{dt} = A_h + \eta_h V_h - (\lambda_h + b + \mu_h)S_h, \\ \frac{dV_h}{dt} = bS_h - \lambda_h V_h - (\eta_h + \mu_h)V_h, \\ \frac{dE_h}{dt} = \lambda_h(S_h + V_h) - (\gamma_h + \mu_h)E_h, \\ \frac{dI_h}{dt} = \gamma_h E_h - (\varepsilon_h + \delta_h + \mu_h)I_h, \\ \frac{dR_h}{dt} = \varepsilon_h I_h - \mu_h R_h, \\ \frac{dL}{dt} = m_1 I_h + m_2 I_r - \mu_l L, \\ \frac{dS_r}{dt} = \frac{p}{\mu_r}(1 - \frac{N_r}{C}) - (\lambda_r + \mu_r)S_r, \\ \frac{dE_r}{dt} = \lambda_r S_r - (\phi_r + \mu_r)E_r, \\ \frac{dI_r}{dt} = \phi_r E_r - \mu_r I_r, \end{cases} \tag{3.1}$$

where $\lambda_h = \beta_h I_h + \beta_{eh}\frac{L}{n+L} + \beta_{rh} I_r,\ \lambda_r = \beta_{hr} I_h + \beta_r I_r$. And with a given initial conditions are $S_h(0) = S_{h0}, V_h(0) = V_0, E_h(0) = E_{h0}, I_h(0) = I_{h0}, R_h(0) = R_{h0}, L(0) = L_0, S_r(0) = S_{r0}, E_r(0) = E_{r0}, I_r(0) = I_{r0}$. Table 1 provides an overview of all the model (3.1) parameters.

We recast the ordinary differential model (3.1) fractional-order system using time-dependent integrals as shown below to account for the memory effect.

$$\begin{cases} \frac{dS_h}{dt} = \int_{t_0}^{t} m_*(t-s)(A_h + \eta_h V_h - (\lambda_h + b + \mu_h)S_h)ds, \\ \frac{dV_h}{dt} = \int_{t_0}^{t} m_*(t-s)(bS_h - \lambda_h V_h - (\eta_h + \mu_h)V_h)ds, \\ \frac{dE_h}{dt} = \int_{t_0}^{t} m_*(t-s)(\lambda_h(S_h + V_h) - (\gamma_h + \mu_h)E_h)ds, \\ \frac{dI_h}{dt} = \int_{t_0}^{t} m_*(t-s)(\gamma_h E_h - (\varepsilon_h + \delta_h + \mu_h)I_h)ds, \\ \frac{dR_h}{dt} = \int_{t_0}^{t} m_*(t-s)(\varepsilon_h I_h - \mu_h R_h)ds, \\ \frac{dL}{dt} = \int_{t_0}^{t} m_*(t-s)(m_1 I_h + m_2 I_r - \mu_l L)ds, \\ \frac{dS_r}{dt} = \int_{t_0}^{t} m_*(t-s)(\frac{p}{\mu_r}(1 - \frac{N_r}{C}) - (\lambda_r + \mu_r)S_r)ds, \\ \frac{dE_r}{dt} = \int_{t_0}^{t} m_*(t-s)(\lambda_r S_r - (\phi_r + \mu_r)E_r)ds, \\ \frac{dI_r}{dt} = \int_{t_0}^{t} m_*(t-s)(\phi_r E_r - \mu_r I_r)ds, \end{cases} \tag{3.2}$$

where $m_*(t-s)$ is the power law correlation function that follows defines a time-dependent kernel.

$$m_*(t-s) = \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)},$$

where $0 < \alpha \leq 1$ and $\Gamma(.)$ represents Gamma function. From here, we simply rewrite (3.2) as

$$\begin{cases} \frac{dS_h}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(A_h + \eta_h V_h - (\lambda_h + b + \mu_h)S_h)ds, \\ \frac{dV_h}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(bS_h - \lambda_h V_h - (\eta_h + \mu_h)V_h)ds, \\ \frac{dE_h}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(\lambda_h(S_h + V_h) - (\gamma_h + \mu_h)E_h)ds, \\ \frac{dI_h}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(\gamma_h E_h - (\varepsilon_h + \delta_h + \mu_h)I_h)ds, \\ \frac{dR_h}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(\varepsilon_h I_h - \mu_h R_h)ds, \\ \frac{dP}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(m_1 I_h + m_2 I_r - \mu_l L)ds, \\ \frac{dS_r}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(\frac{p}{\mu_r}(1 - \frac{N_r}{C}) - (\lambda_r + \mu_r)S_r)ds, \\ \frac{dE_r}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(\lambda_r S_r - (\phi_r + \mu_r)E_r)ds, \\ \frac{dI_r}{dt} = \frac{1}{\Gamma(\alpha-1)} \int_{t_0}^{t}(t-s)^{\alpha-2}(\phi_r E_r - \mu_r I_r)ds. \end{cases} \tag{3.3}$$

From (3.3), we can see that the right-hand side is fractional integral of order $(\alpha - 2), 0 < \alpha \leq 1$ on the interval $[t, t_0]$ as defined by ${}_{t_0}D_t^{-(\alpha-1)}$. We apply Caputo fractional derivative of order $(\alpha - 1)$ on both sides of (3.3) because we are aware that fractional derivatives are the left inverse of fractional integrals. In doing so, our proposed model in the Caputo derivative is obtained. Thus;

$$\begin{cases} {}^{C}D_t^{\alpha} S_h(t) = A_h + \eta_h V_h - (\lambda_h + b + \mu_h) S_h, \\ {}^{C}D_t^{\alpha} V_h(t) = b S_h - \lambda_h V_h - (\eta_h + \mu_h) V_h, \\ {}^{C}D_t^{\alpha} E_h(t) = \lambda_h (S_h + V_h) - (\gamma_h + \mu_h) E_h, \\ {}^{C}D_t^{\alpha} I_h(t) = \gamma_h E_h - (\varepsilon_h + \delta_h + \mu_h) I_h, \\ {}^{C}D_t^{\alpha} R_h(t) = \varepsilon_h I_h - \mu_h R_h, \\ {}^{C}D_t^{\alpha} L(t) = m_1 I_h + m_2 I_r - \mu_l L, \\ {}^{C}D_t^{\alpha} S_r(t) = \frac{p}{\mu_r}(1 - \frac{N_r}{C}) - (\lambda_r + \mu_r) S_r, \\ {}^{C}D_t^{\alpha} E_r(t) = \lambda_r S_r - (\phi_r + \mu_r) E_r, \\ {}^{C}D_t^{\alpha} I_r(t) = \phi_r E_r - \mu_r I_r, \end{cases} \tag{3.4}$$

with the following initial conditions $S_h(0) = S_{h_0} \geq 0,\ V_h(0) = V_{h_0} \geq 0,\ E_h(0) = E_{h_0} \geq 0,\ I_h(0) = I_{h_0} \geq 0, R_h(0) = R_{h_0} \geq 0, L(0) = L_0 \geq 0,\ S_r(0) = S_{r_0} \geq 0,\ E_r(0) = E_{r_0} \geq 0,\ I_r(0) = I_{r_0} \geq 0$.

Table 1. Parameter description and suitable values for the model.

Parameter	Description	Value
A_h	Recruitment into susceptible human class.	450
η_h	Efficacy loss rate	0.0034
μ_h	Rates of human natural death	0.00067
μ_r	Rates of rodents natural death	0.004
μ_l	The natural death rate of pathogens	0.08
b	The rate of vaccine and efficacy rate	0.4
C	Rodent carrying capacity	0.01
γ_h	The rate at which humans who are exposed join infection class	0.005
ϕ_r	The rate at which exposed rodents join infection class	0.005
ε_h	The rate at which humans recovery from monkey pox	0.18
δ_h	Disease-induced mortality rate for humans	0.0013
β_h	Transmission rates from infected persons to susceptible humans	0.0001
m_1	Rates of viral contribution from human to environment	0.0069
m_2	Rates of viral contribution from rodent to environment	0.0051
p	Rodent maximum growth rate	91
β_{eh}	Transmission rates of environment to susceptible population of humans	0.0061
β_{rh}	Transmission rates for susceptible humans from infected rodents	0.000041
β_{hr}	Transmission rates from the infected humans to rodents that are susceptible	0.000041
β_r	Rates of infection spread from infected rodents to susceptible rodent	0.000009

4. Analysis of the model

4.1. Basic mathematical properties of the model

We carried out the qualitative analysis of the models (3.1) and (3.4). Since (3.1) and (3.4) depict population of both humans and rodents during an outbreak of the MPXV, the epidemiological significant of the state variables are positive for $t \geq 0$. This means that the proposed model solution for all times $t \geq 0$ with positive beginning data will remain positive.

Theorem 1. Suppose that $\{(S_{h_0}, V_{h_0}, E_{h_0}, I_{h_0}, R_{h_0}, L_0, S_{r_0}, E_{r_0}, I_{r_0}) \in R\}, \exists\ t_0$ and continuous function $(S_h(t), V_h(t), E_h(t), I_h(t), R_h(t), L(t), S_v(t), E_v(t), I_v(t) : [0, t_0) \to R)$ such that
$S_h(t), V_h(t), E_h(t), I_h(t), R_h(t), L(t), S_v(t), E_v(t), I_v(t)$ satisfies our proposed MPOX model and

$$(S_h, V_h, E, I_h, R_h, L, S_r, E_r, I_r)(0) = (S_{h_0}, V_{h_0}, E_{h_0}, I_{h_0}, R_{h_0}, L_0, S_{r_0}, E_{r_0}, I_{r_0}).$$

Remark 2. If the unique solution obtained from Theorem 1 for the interval $[0, t_0]$, there exists $t_0 > 0$ then, the function $S_h(t), V_h(t), E_h(t), I_h(t), R_h(t), L(t), S_v(t), E_v(t)$, and $I_v(t)$ will be bounded and for all $t \in [0, t_0]$ remain positive.

Proof. Using the Classical Picard-Lindelof theorem and following the discussion of [37], we conclude that Theorem 1 and Remark 2 hold. Hence we omit the proof.

Theorem 3. The solution of the proposed MPOX model is enclosed in the region $\mathbf{D}_*$ subset R^9_+, given by

$$\mathbf{D}_* = \{(S_h, V_h, E_h, I_h, R_h, L, S_r, E_r, I_r) \in R^9_+ : N_h(t) \leq \frac{A_h}{\mu_h},\ N_r(t) \leq (\frac{p}{\mu_r})\},$$

for the initial conditions (3.1) in $\mathbf{D}_*$.

Proof. We know that

$$\begin{cases} \frac{dN_h(t)}{dt} = A_h - \mu_h N_h(t), \\ \frac{dN_r(t)}{dt} = p - (\frac{p}{C} + \mu_r)N_r(t) \geq p - \mu_r N_r(t). \end{cases} \tag{4.1}$$

Thus, $\frac{dN_h(t)}{dt} < 0; \frac{dN_r(t)}{dt} < 0$, if $N_h(t) > \frac{A_h}{\mu_h}$ and $N_r(t) > \frac{p}{\mu_r}$.

By a standard comparison theorem, we have

$$N_h(t) \leq N_h(0)e^{-\mu_h(t)} + \frac{A_h}{\mu_h}(1 - e^{-\mu_h(t)}),\ N_r(t) \leq N_r(0)e^{-\mu_r(t)} + (\frac{p}{\mu_r})(1 - e^{-\mu_r(t)}).$$

And we have $N_h(t) \leq \frac{A_h}{\mu_h}$ and $N_r(t) \leq \frac{p}{\mu_r}$ if $N_h(0) \leq \frac{A_h}{\mu_h}$ and $N_r(0) \leq \frac{p}{\mu_r}$, respectively. Then, $\mathbf{D}_*$ is invariant positively. Further, $N_h(t) > \frac{A_h}{\mu_h}$ and $N_r(t) > \frac{p}{\mu_r}$, then solution enters $\mathbf{D}_*$ infinite time. Thus, $\mathbf{D}_*$ attracts all solutions in R^9_+. Hence, the model (3.1) is epidemiologically well-posed in $\mathbf{D}_*$.

Theorem 4. Under initial conditions, the solution of proposed system (3.4) is nonnegative and bounded in R^9_+. Therefore

$$\begin{cases} \lim_{t\to\infty}\sup S_h(t) \le S_{h\infty} = \frac{A_h+\eta_h V_{h\infty}}{\lambda_h+b+\mu_h}, \\ \lim_{t\to\infty}\sup V_h(t) \le V_{h\infty} = \frac{bS_{h\infty}}{\lambda_h+\eta_h+\mu_h}, \\ \lim_{t\to\infty}\sup E_h(t) \le E_{h\infty} = \frac{\lambda_h(V_{h\infty}+S_{h\infty})}{\gamma_h+\mu_h}, \\ \lim_{t\to\infty}\sup I_h(t) \le I_{h\infty} = \frac{\gamma_h E_{h\infty}}{\varepsilon_h+\delta_h+\mu_h}, \\ \lim_{t\to\infty}\sup R_h(t) \le R_{h\infty} = \frac{\varepsilon_h I_{h\infty}}{\mu_h}, \\ \lim_{t\to\infty}\sup L(t) \le L_{\infty} = \frac{m_1 I_{h\infty}+m_2 I_{r\infty}}{\mu_h}, \\ \lim_{t\to\infty}\sup S_r(t) \le S_{r\infty} = \frac{\frac{p}{\mu_r}(1-\frac{N_r}{C})}{\lambda_r+\mu_r}, \\ \lim_{t\to\infty}\sup E_r(t) \le E_{r\infty} = \frac{\lambda_r S_{r\infty}}{\phi_r+\mu_r}, \\ \lim_{t\to\infty}\sup I_r(t) \le I_{r\infty} = \frac{\phi_r E_{r\infty}}{\mu_r}. \end{cases} \tag{4.2}$$

Proof. Using the knowledge in [38] and with the initial values provided, we derive from model (3.4) as

$$\begin{cases} {}^C D_t^\alpha S_h(t) = A_h + \eta_h V_h > 0, \\ {}^C D_t^\alpha V_h(t) = bS_h - \lambda_h V_h > 0, \\ {}^C D_t^\alpha E_h(t) = \lambda_h(S_h + V_h) > 0, \\ {}^C D_t^\alpha I_h(t) = \gamma_h E_h > 0, \\ {}^C D_t^\alpha R_h(t) = \varepsilon_h I_h > 0, \\ {}^C D_t^\alpha L(t) = m_1 I_h + m_2 I_r > 0, \\ {}^C D_t^\alpha S_r(t) = \frac{p}{\mu_r}(1 - \frac{N_r}{C}) > 0, \\ {}^C D_t^\alpha E_r(t) = \lambda_r S_r > 0, \\ {}^C D_t^\alpha I_r(t) = \phi_r E_r > 0. \end{cases} \tag{4.3}$$

From (4.3), the result cannot escape from the hyperplanes, since $S_h(0) > 0, V_h(0) > 0, E_h(0) > 0, I_h(0) > 0, R_h(0) > 0, L(0) > 0, S_r(0) > 0, E_r(0) > 0, I_r(0) > 0$, for all $t > 0$. Hence, model (3.4) is nonnegative and bounded.

4.2. *Disease free equilibrium (DFE)*

In order to determine the equilibrium level at which the epidemic completely eliminated the population. Setting $E_h = I_h = L = E_r = I_R = 0$ and reducing the right hand side of the model equation to zero, leads to;

$$\begin{cases} A_h + \eta_h V_{h0} - (b + \mu_h)S_{h0} = 0, \\ bS_h - \lambda_h S_{h0} - (\eta_h + \mu_h)V_{h0} = 0, \\ p(1 - \frac{N_r}{C}) - (+\mu_r)S_{r0} = 0. \end{cases} \tag{4.4}$$

Then by rearranging model (4.4) and after the manipulation we obtain ,

$$\begin{cases} S_{h0} = \frac{A_h}{(b+\mu_h)}, \\ S_{h0} = \frac{bA_h}{(b+\mu_h)(\eta_h+\mu_h)} = \frac{A_h\rho}{(b+\mu_h)}, \\ S_{r0} = p(1 - \frac{N_r}{\mu_r C}), \end{cases} \tag{4.5}$$

where $\rho = \frac{b}{(\eta_1+\mu_h)}$. Hence the disease DFE point of our proposed model is given as

$$E_0 = (S_h, V_h, E_h, I_h, R_h, L, S_r, E_r, I_r) = \Big(\frac{A_h}{(b+\mu_h)},\ \ \frac{A_h\rho}{(b+\mu_h)},\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ p(1 - \frac{N_r}{\mu_r C}),\ \ 0,\ \ 0\Big). \tag{4.6}$$

4.3. Basic reproduction number

The average number of secondary infections that an infected person causes during his infectious period is the basic reproduction number. Using the next-generation operator approach in a dynamical system with the rate of emergence of new infections $\mathbb{F}$ and the rate of individual transmission $\mathbb{V}$ at the steady-state disease-free condition, this is calculated as

$$E_0 = \left(\tfrac{A_h}{(b+\mu_h)}, \quad \tfrac{A_h\rho}{(b+\mu_h)}, \quad 0, \quad 0, \quad 0, \quad 0, \quad p(1-\tfrac{N_r}{\mu_r C}), \quad 0, \quad 0\right) \tag{4.7}$$

is

$$\mathbb{F} = \begin{pmatrix} 0 & \beta_h\frac{A_h(1+\rho)}{(b+\mu_h)} & \beta_{eh}\frac{A_h(1+\rho)}{(b+\mu_h)} & 0 & \beta_{rh}\frac{A_h(1+\rho)}{(b+\mu_h)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p(1-\frac{N_r}{\mu_r C})(\beta_{hr}+\beta_r) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.8}$$

$$\mathbb{V} = \begin{pmatrix} \gamma_h+\mu_h & 0 & 0 & 0 & 0 \\ \gamma_h & \varepsilon_h+\delta_h+\mu_h & 0 & 0 & 0 \\ 0 & -m_1 & \mu_l & -m_2 & 0 \\ 0 & 0 & 0 & \phi_r+\mu_r & 0 \\ 0 & 0 & 0 & -\phi_r & \mu_r \end{pmatrix}, \tag{4.9}$$

$$\mathbb{V}^{-1} = \begin{pmatrix} \frac{1}{\gamma_h+\mu_h} & 0 & 0 & 0 & 0 \\ \frac{\gamma_h}{(\gamma_h+\mu_h)(\varepsilon_h+\delta_h+\mu_h)} & \frac{1}{\varepsilon_h\delta_h+\mu_h} & 0 & 0 & 0 \\ \frac{m_1\gamma_h}{(\gamma_h+\mu_h)(\varepsilon_h+\delta_h+\mu_h)} & \frac{m_1}{(\gamma_h+\mu_h)\mu_l} & \frac{1}{\mu_l} & \frac{m_2}{\phi_r+\mu_r} & 0 \\ 0 & 0 & 0 & \frac{1}{\phi_r+\mu_r} & 0 \\ 0 & 0 & 0 & \frac{\phi_r}{(\phi_r+\mu_r)\mu_l} & \frac{1}{\mu_r} \end{pmatrix}. \tag{4.10}$$

Thus, the basic reproduction number $\Re$ is the spectral radius of $\mathbb{F}\mathbb{V}^{-1}$

$$\Re = \frac{A_h(1+\rho)\gamma_r(m_1\beta_{eh}+\beta_r\mu_l)}{(b+\mu_h)(\gamma_h+\mu_h)(\varepsilon_h+\delta_h+\mu_h)\mu_l}$$

4.4. Existence and stability of the endemic equilibrium point

We denote the endemic equilibrium of the MPOX model as $(S_h^*, V_h^*, E_h^*, I_h^*, R_h^*, L^*, S_r^*, E_r^*, I_r^*)$ such that

$$S_h^* = \frac{A_h+\eta_h V_h^*}{\lambda_h^*+b+\mu_h}; \; V_h^* = \frac{bS_h^*}{\lambda_h^*+\eta_h+\mu_h}; \; E_h^* = \frac{\lambda_h^*(V_h^*+S_h^*)}{\gamma_h+\mu_h}$$

$$I_h^* = \frac{\gamma_h E_h^*}{\varepsilon_h+\delta_h+\mu_h} = \frac{\gamma_h\lambda_h^*(V_h^*+S_h^*)}{(\varepsilon_h+\delta_h+\mu_h)(\gamma_h+\mu_h)} = p_1\lambda_h^*(V_h^*+S_h^*);$$

$$R_h^* = \frac{\varepsilon_h I_h^*}{\mu_h}; \; L^* = \frac{m_1 I_h^*+m_2 I_r^*}{\mu_h} = \frac{m_1 p_1\lambda_h^*(V_h^*+S_h^*)+m_2 p_2 E_r^*}{\mu_l}$$

$$S_r^* = \frac{p(1-\frac{N_r}{C})}{\lambda_r^*+\mu_r}; \; E_r^* = \frac{\lambda_r^* S_r^*}{\phi_r+\mu_r}; \; I_r^* = \frac{\phi_r E_r^*}{\mu_r} = p_2 E_r^*,$$

where $\lambda_h^* = \beta_h I_h^* + \beta_{eh}\frac{L^*}{n+L^*} + \beta_{rh} I_r^*,\ \lambda_r^* = \beta_{hr} I_h^* + \beta_r I_r^*, p_1 = \frac{\gamma_h}{(\varepsilon_h+\delta_h+\mu_h)(\gamma_h+\mu_h)}, p_2 = \frac{\phi_r}{\mu_r}$. Further, we have

$$\lambda_h^* = \beta_h p_1 \lambda_h^*(V_h^* + S_h^*) + \beta_{eh}\frac{m_1 p_1 \lambda_h^*(V_h^* + S_h^*) + m_2 p_2 E_r^*}{n\mu_l + m_1 p_1 \lambda_h^*(V_h^* + S_h^*) + m_2 p_2 E_r^*} + \beta_{rh} p_2 E_r^*,$$

upon further simplification, we have

$$(\beta_h p_1 \lambda_h^* + \beta_{eh} m_1 p_1 \lambda_h^* - m_1 p_1 \lambda_h^{*2})(V_h^* + S_h^*) + (\beta_{eh} m_2 p_2 + \beta_{rh} p_1 - m_2 p_2 \lambda_h^*)E_r^* - \mu_l n \lambda_h^* = 0,$$

and

$$\lambda_r^* = \beta_{hr} p_1 \lambda_h^*(V_h^* + S_h^*) + \beta_r p_2 E_r^*,$$

but $(V_h^* + S_h^*) = \frac{E_h^*(\gamma_h+\mu_h)}{\lambda_h^*}$. By substitution, we have

$$(\beta_h p_1 + \beta_{eh} m_1 p_1 - m_1 p_1 \lambda_h^*)E_h^*(\gamma_h + \mu_h) + (\beta_{eh} m_2 p_2 + \beta_{rh} p_1 - m_2 p_2 \lambda_h^*)E_r^* - \mu_l n \lambda_h^* = 0,$$

so that

$$\lambda_r^* = \beta_{hr} p_1 E_h^*(\gamma_h + \mu_h) + \beta_r p_2 E_r^* > 0,$$

whenever $\lambda_h^* > 0$. Therefore, the existence of endemic equilibrium point is obtained by substituting the unique values of λ_h^* and λ_r^*.

Theorem 5. Suppose that $\Re > 1$, then endemic equilibrium E_{**} of the model (3.1) is globally asymptotically stable.

Proof. We employ the Lyapunov approach to demonstrate the endemic equilibrium's asymptotically global stability and unstable if $\Re < 1$. We consider Lyapunov function

$$\begin{aligned}\mathcal{F}(S_h^*, V_h^*, E_h^*, I_h^*, R_h^*, L^*, S_r^*, E_r^*, I_r^*) &= (S_h - S_h^* - S_h^* \ln\frac{S_h}{S_h^*}) + (V_h - V_h^* - V_h^* \ln\frac{V_h}{V_h^*}) \\ &+ (E_h - E_h^* - E_h^* \ln\frac{E_h}{E_h^*}) + (I_h - I_h^* - I_h^* \ln\frac{I_h}{I_h^*}) \\ &+ (R_h - R_h^* - R_h^* \ln\frac{R_h}{R_h^*}) + (L - L^* - L^* \ln\frac{L}{L^*}) \\ &+ (S_r - S_r^* - S_r^* \ln\frac{S_r}{S_r^*})(E_r - E_r^* - E_r^* \ln\frac{E_r}{E_r^*}) \\ &+ (I_r - I_r^* - I_r^* \ln\frac{I_r}{I_r^*}).\end{aligned} \tag{4.11}$$

Now find the derivative of $\mathbb{F}$

$$\begin{aligned}\frac{d\mathcal{F}}{dt} &= (1 - \frac{S_h^*}{S_h})\frac{dS_h}{dt} + (1 - \frac{V_h^*}{V_h})\frac{dV_h}{dt} + (1 - \frac{E_h^*}{E_h})\frac{dE_h}{dt} + (1 - \frac{I_h^*}{I_h})\frac{dI_h}{dt} + (1 - \frac{R_h^*}{R_h})\frac{dR_h}{dt} \\ &+ (1 - \frac{L^*}{L})\frac{dL}{dt} + (1 - \frac{S_r^*}{S_r})\frac{dS_r}{dt} + (1 - \frac{E_r^*}{E_r})\frac{dE_r}{dt} + (1 - \frac{I_r^*}{I_r})\frac{dI_r}{dt}.\end{aligned} \tag{4.12}$$

Substituting the value of $\frac{dS_h}{dt}, \frac{dV_h}{dt}, \frac{dE_h}{dt}, \frac{dI_h}{dt}, \frac{dR_h}{dt}, \frac{dL}{dt}, \frac{dS_r}{dt}, \frac{dE_r}{dt}, \frac{dI_r}{dt}$ from the model, the following is obtain

$$\frac{d\mathcal{F}}{dt} = (1-\frac{S_h^*}{S_h})[A_h+\eta_h V_h-(\lambda_h+b+\mu_h)S_h]+(1-\frac{V_h^*}{V_h})[bS_h-\lambda_h V_h-(\eta_h+\mu_h)V_h]$$
$$+(1-\frac{E_h^*}{E_h})[\lambda_h(S_h+V_h)-(\gamma_h+\mu_h)E_h]+(1-\frac{I_h^*}{I_h})[\gamma_h E_h-(\varepsilon_h+\delta_h$$
$$+\mu_h)I_h]+(1-\frac{R_h^*}{R_h})[\varepsilon_h I_h-\mu_h R_h]+(1-\frac{L^*}{L})[m_1 I_h+m_2 I_r-\mu_l L]$$
$$+(1-\frac{S_r^*}{S_r})[\frac{p}{\mu_r}(1-\frac{N_r}{C})-(\lambda_r+\mu_r)S_r]+(1-\frac{E_r^*}{E_r})[\lambda_r S_r-(\phi_r+\mu_r)E_r]+(1-\frac{I_r^*}{I_r})[\phi_r E_r-\mu_r I_r]. \tag{4.13}$$

At the endemic point of model (3.1), we obtain

$$(\gamma_h+\mu_h)=\frac{\lambda_h(V_h+S_h)}{E_h^*},\ (\varepsilon_h+\delta_h+\mu_h)=\frac{\gamma_h E_h^*}{I_h^*},\ \mu_h=\frac{\varepsilon_h I_h^*}{R_h^*},$$
$$(\lambda_r+\mu_r)=\frac{\frac{p}{\mu_r}(1-\frac{N_r}{C})}{S_r^*},\ (\phi_r+\mu_r)=\frac{\lambda_r S_r^*}{E_r^*},\ \mu_r=\frac{\phi_r E_r^*}{I_r^*}. \tag{4.14}$$

Substituting (4.14) into (4.13), we obtain

$$\frac{d\mathcal{F}}{dt} = (1-\frac{S_h^*}{S_h})[A_h+\eta_h V_h-(\lambda_h+b+\mu_h)S_h]+(1-\frac{V_h^*}{V_h})[bS_h-\lambda_h V_h-(\eta_h+\mu_h)V_h]$$
$$+(1-\frac{E_h^*}{E_h})[\lambda_h(S_h+V_h)-(\frac{\lambda_h(V_h+S_h)}{E_h^*})E_h]+(1-\frac{I_h^*}{I_h})[\gamma_h E_h-(\frac{\gamma_h E_h^*}{I_h^*})I_h]$$
$$+(1-\frac{R_h^*}{R_h})[\varepsilon_h I_h-(\frac{\varepsilon_h I_h^*}{R_h^*})R_h]+(1-\frac{L^*}{L})[m_1 I_h+m_2 I_r-\mu_l L] \tag{4.15}$$
$$+(1-\frac{S_r^*}{S_r})[\frac{p}{\mu_r}(1-\frac{N_r}{C})-(\frac{\frac{p}{\mu_r}(1-\frac{N_r}{C})}{S_r^*})S_r]+(1-\frac{E_r^*}{E_r})[\lambda_r S_r-(\frac{\lambda_r S_r^*}{E_r^*})E_r]$$
$$+(1-\frac{I_r^*}{I_r})[\phi_r E_r-\frac{\phi_r E_r^*}{I_r^*}I_r].$$

Let $S_h = S_h^*, V_h = V_h^*, E_h = E_h^*, I_h = I_h^*, R_h = R_h^*, L = L^*, S_r = S_r^*, E_r = E_r^*, I_r = I_r^*$, then $\frac{d\mathcal{F}}{dt} = 0$. Hence by LaSalles invariance principle every solution of model (3.1) with initial condition in $\Pi = \{(S_h, V_h, E_h, I_h, R_h, L, S_r, E_r, I_r) \in R_+^9\}$ it follows that E_{**} is globally asymptotically stable.

Theorem 6. Suppose that $\Re < 1$, then endemic equilibrium E_{**} of the model (3.1) is locally asymptotically stable.

The disease-free equilibrium point is locally stable and the population cannot be infected by the disease if the reproduction number is less than one. Applying the proof from Theorem 2 of [39] the proof of the above theorem is therefore valid.

5. Existence and uniqueness solution for Caputo model

In this section the existence and uniqueness solutions for Caputo model will be provided herein for the (3.4). Supposed that a continuous real-value function denoted $\mathtt{W(J)}$ containing the sup norm property is a Banach space on $\mathtt{J} = [0,\eta]$ and $\mathbb{M} = \mathtt{W(J)}\times\mathtt{W(J)}\times\mathtt{W(J)}\times\mathtt{W(J)}\times\mathtt{W(J)}\times\mathtt{W(J)}\times\mathtt{W(J)}\mathtt{W(J)}\times$

W(J) with norm $\|(S_h, V_h, E_h, I_h, R_h, L, S_r, E_r, I_r)\| = \|S_h\| + \|V_h\| + \|E_h\| + \|I_h\| + \|R_h + \|L\| + \|S_r\| + \|E_r\| + \|I_r\|$, where $\|S_h\| = \sup_{t\in J}|S_h|, \|V_h\| = \sup_{t\in J}|V_h|, \|E_h\| = \sup_{t\in J}|E_h|, \|I_h\| = \sup_{t\in J}|I_h|, \|R_h\| = \sup_{t\in J}|R_h|, \|L\| = \sup_{t\in J}|L|, \|S_r\| = \sup_{t\in J}|S_r|, \|E_r\| = \sup_{t\in J}|E_r|, \|I_r\| = \sup_{t\in J}|I_r|$. Now when we apply the Caputo fractional integral operator to the both sides of (3.4) we obtain

$$\begin{cases} S_h(t) - S_h(0) = {}^C D_t^\alpha\{A_h + \eta_h V_h - (\lambda_h + b + \mu_h)S_h\}, \\ V_h(t) - V_h(0) = {}^C D_t^\alpha\{bS_h - \lambda_h V_h - (\eta_h + \mu_h)V_h\}, \\ E_h(t) - E_h(0) = {}^C D_t^\alpha\{\lambda_h(S_h + V_h) - (\gamma_h + \mu_h)E_h\}, \\ I_h(t) - I_h(0) = {}^C D_t^\alpha\{\gamma_h E_h - (\varepsilon_h + \delta_h + \mu_h)I_h\}, \\ R_h(t) - R_h(0) = {}^C D_t^\alpha\{\varepsilon_h I_h - \mu_h R_h\}, \\ L(t) - L(0) = {}^C D_t^\alpha\{m_1 I_h + m_2 I_r - \mu_l L\}, \\ S_r(t) - S_r(0) = {}^C D_t^\alpha\{\frac{p}{\mu_r}(1 - \frac{N_r}{C}) - (\lambda_r + \mu_r)S_r\}, \\ E_r(t) - E_r(0) = {}^C D_t^\alpha\{\lambda_r S_r - (\phi_r + \mu_r)E_r\}, \\ I_r(t) - I_r(0) = {}^C D_t^\alpha\{\phi_r E_r - \mu_r I_r\}. \end{cases} \tag{5.1}$$

And this then lead to

$$\begin{cases} S_h(t) - S_h(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_1(\alpha, \Theta, S_h(\Theta))d\Theta, \\ V_h(t) - V_h(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_2(\alpha, \Theta, V_h(\Theta))d\Theta, \\ E_h(t) - E_h(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_3(\alpha, \Theta, E_h(\Theta))d\Theta, \\ I_h(t) - I_h(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_4(\alpha, \Theta, I_h(\Theta))d\Theta, \\ R_h(t) - R_h(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_5(\alpha, \Theta, R_h(\Theta))d\Theta, \\ L(t) - L(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_6(\alpha, \Theta, L(\Theta))d\Theta, \\ S_r(t) - S_r(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_7(\alpha, \Theta, S_r(\Theta))d\Theta, \\ E_r(t) - E_r(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_8(\alpha, \Theta, E_r(\Theta))d\Theta, \\ I_r(t) - I_r(0) = \mathbb{H}(\alpha)\int_0^t (t-\Theta)^{-\alpha}\Upsilon_9(\alpha, \Theta, I_r(\Theta))d\Theta, \end{cases} \tag{5.2}$$

supposing that

$$\begin{cases} \Upsilon_1(\alpha, t, S_h(t)) = A_h + \eta_h V_h - (\lambda_h + b + \mu_h)S_h, \\ \Upsilon_2(\alpha, t, V_h(t)) = bS_h - \lambda_h V_h - (\eta_h + \mu_h)V_h, \\ \Upsilon_3(\alpha, t, E_h(t)) = \lambda_h(S_h + V_h) - (\gamma_h + \mu_h)E_h, \\ \Upsilon_4(\alpha, t, I_h(t)) = \gamma_h E_h - (\varepsilon_h + \delta_h + \mu_h)I_h, \\ \Upsilon_5(\alpha, t, R_h(t)) = \varepsilon_h I_h - \mu_h R_h, \\ \Upsilon_6(\alpha, t, L(t)) = m_1 I_h + m_2 I_r - \mu_l L, \\ \Upsilon_7(\alpha, t, S_r(t)) = \frac{p}{\mu_r}(1 - \frac{N_r}{C}) - (\lambda_r + \mu_r)S_r, \\ \Upsilon_8(\alpha, t, E_r(t)) = \lambda_r S_r - (\phi_r + \mu_r)E_r, \\ \Upsilon_9(\alpha, t, I_r(t)) = \phi_r E_r - \mu_r I_r. \end{cases} \tag{5.3}$$

Now we note that Υ_i where $i = 1, 2, 3, ..., 9$ satisfies the Lipschitz condition if and only if $S_h(t)$, $V_h(t)$, $E_h(t), I_h(t)$, $R_h(t)$, $L(t)$, $S_v(t)$, $E_v(t)$, and $I_v(t)$ are upper bound. Surmising that $S_h(t)$ and $\mathbb{S}_h^{**}(t)$ are two function, so we have

$$\begin{aligned} \|\Upsilon_1(\alpha, t, S_h(t)) - \Upsilon_1(\alpha, t, \mathbb{S}_h(t))\| &= \|(\lambda_h + b + \mu_h)(S_h - \mathbb{S}_h^{**})\|, \\ &= (\lambda_h + b + \mu_h)\|S_h - \mathbb{S}_h^{**})\|. \end{aligned} \tag{5.4}$$

Therefore, taking $\zeta_1 = (\lambda_h + b + \mu_h)$, we have

$$\|\Upsilon_1(\alpha, t, S_h(t)) - \Upsilon_1(\alpha, t, \mathbb{S}_h(t))\| \leq \zeta_1 \|S_h - \mathbb{S}_h^{**})\|, \quad (5.5)$$

similarly, we obtain

$$\begin{aligned}
&\|\Upsilon_2(\alpha, t, V_h(t)) - \Upsilon_2(\alpha, t, \mathbb{V}_h(t))\| \leq \zeta_2 \|V_h - \mathbb{V}_h^{**})\|,\\
&\|\Upsilon_3(\alpha, t, E_h(t)) - \Upsilon_3(\alpha, t, \mathbb{E}_h(t))\| \leq \zeta_3 \|E_h - \mathbb{E}_h^{**})\|,\\
&\|\Upsilon_4(\alpha, t, I_h(t)) - \Upsilon_4(\alpha, t, \mathbb{I}_h(t))\| \leq \zeta_4 \|I_h - \mathbb{I}_h^{**})\|,\\
&\|\Upsilon_5(\alpha, t, R_h(t)) - \Upsilon_5(\alpha, t, \mathbb{R}_h(t))\| \leq \zeta_5 \|R_h - \mathbb{R}_h^{**})\|,\\
&\|\Upsilon_6(\alpha, t, L(t)) - \Upsilon_6(\alpha, t, \mathbb{L}(t))\| \leq \zeta_6 \|L - \mathbb{L}^{**})\|,\\
&\|\Upsilon_7(\alpha, t, S_r(t)) - \Upsilon_7(\alpha, t, \mathbb{S}_r(t))\| \leq \zeta_7 \|S_r - \mathbb{S}_r^{**})\|,\\
&\|\Upsilon_8(\alpha, t, E_r(t)) - \Upsilon_8(\alpha, t, \mathbb{E}_r(t))\| \leq \zeta_8 \|E_r - \mathbb{E}_r^{**})\|,\\
&\|\Upsilon_9(\alpha, t, I_r(t)) - \Upsilon_9(\alpha, t, \mathbb{I}_r(t))\| \leq \zeta_9 \|I_r - \mathbb{I}_r^{**})\|.
\end{aligned} \quad (5.6)$$

This indicates that the Lipschitz condition is fulfilled for Υ_i where $i = 1, 2, 3, ..., 9$ Recursively, the model (5.1) yield

$$\begin{cases}
S_{h_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_1(\alpha, \Theta, S_{h_{n-1}}(\Theta)) d\Theta,\\
V_{h_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_2(\alpha, \Theta, V_{h_{n-1}}(\Theta)) d\Theta,\\
E_{h_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_3(\alpha, \Theta, E_{h_{n-1}}(\Theta)) d\Theta,\\
I_{h_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_4(\alpha, \Theta, I_{h_{n-1}}(\Theta)) d\Theta,\\
R_{r_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_5(\alpha, \Theta, R_{h_{n-1}}(\Theta)) d\Theta,\\
L_n(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_6(\alpha, \Theta, L_{n-1}(\Theta)) d\Theta,\\
S_{r_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_7(\alpha, \Theta, S_{r_{n-1}}(\Theta)) d\Theta,\\
E_{r_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_8(\alpha, \Theta, E_{r_{n-1}}(\Theta)) d\Theta,\\
I_{r_n}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} \Upsilon_9(\alpha, \Theta, I_{r_{n-1}}(\Theta)) d\Theta,
\end{cases} \quad (5.7)$$

with the initial conditions; $S_h(0), V_h(0), E_h(0), I_h(0), R_h(0), L(0), S_r(0), E_r(0), I_r(0)$. Taking the difference in the successive terms, we have

$$\begin{cases}
\Psi_{S_{h,n}} = S_{h_n}(t) - S_{h_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_1(\alpha, \Theta, S_{h_{n-1}}(\Theta)) - \Upsilon_1(\alpha, \Theta, S_{h_{n-2}}(\Theta))] d\Theta,\\
\Psi_{V_{h,n}} = V_h(t) - V_{h_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_2(\alpha, \Theta, V_{h_{n-1}}(\Theta)) - \Upsilon_2(\alpha, \Theta, V_{h_{n-2}}(\Theta))] d\Theta,\\
\Psi_{E_{h,n}} = E_h(t) - E_{h_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_3(\alpha, \Theta, E_{h_{n-1}}(\Theta)) - \Upsilon_3(\alpha, \Theta, E_{h_{n-2}}(\Theta))] d\Theta,\\
\Psi_{I_{h,n}} = I_h(t) - I_{h_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_4(\alpha, \Theta, I_{h_{n-1}}(\Theta)) - \Upsilon_4(\alpha, \Theta, I_{h_{n-2}}(\Theta))] d\Theta,\\
\Psi_{R_{h,n}} = R_h(t) - R_{h_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_5(\alpha, \Theta, R_{h_{n-1}}(\Theta)) - \Upsilon_5(\alpha, \Theta, R_{h_{n-2}}(\Theta))] d\Theta,\\
\Psi_{L_n} = L(t) - L_{n-1}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_6(\alpha, \Theta, L_{n-1}(\Theta)) - \Upsilon_6(\alpha, \Theta, L_{n-2}(\Theta))] d\Theta,\\
\Psi_{S_{r,n}} = S_r(t) - S_{r_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_7(\alpha, \Theta, S_{r_{n-1}}(\Theta)) - \Upsilon_7(\alpha, \Theta, S_{r_{n-2}}(\Theta))] d\Theta,\\
\Psi_{E_{r,n}} = E_r(t) - E_{r_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_8(\alpha, \Theta, E_{r_{n-1}}(\Theta)) - \Upsilon_8(\alpha, \Theta, E_{r_{n-2}}(\Theta))] d\Theta,\\
\Psi_{I_{r,n}} = I_r(t) - I_{r_{n-1}}(t) = \mathbb{H}(\alpha) \int_0^t (t-\Theta)^{-\alpha} [\Upsilon_9(\alpha, \Theta, I_{r_{n-1}}(\Theta)) - \Upsilon_9(\alpha, \Theta, I_{r_{n-2}}(\Theta))] d\Theta.
\end{cases} \quad (5.8)$$

Based on the above (5.8), we can see that

$$\begin{aligned}
S_{h_n}(t) = \Sigma_{j=0}^n \Psi_{S_{h,n}}(t),\ V_{h_n}(t) = \Sigma_{j=0}^n \Psi_{V_{h,n}}(t),\ E_{h_n}(t) = \Sigma_{j=0}^n \Psi_{E_{h,n}}(t),\\
I_{h_n}(t) = \Sigma_{j=0}^n \Psi_{I_{h,n}}(t),\ R_{h_n}(t) = \Sigma_{j=0}^n \Psi_{R_{h,n}}(t),\ L_n(t) = \Sigma_{j=0}^n \Psi_{L_n}(t),\\
S_{r_n}(t) = \Sigma_{j=0}^n \Psi_{S_{r,n}}(t),\ E_{r_n}(t) = \Sigma_{j=0}^n \Psi_{E_{r,n}}(t),\ I_{r_n}(t) = \Sigma_{j=0}^n \Psi_{i_{r,n}}(t).
\end{aligned} \quad (5.9)$$

In addition, by using (5.5) and (5.6), and considering

$$\begin{aligned} \Psi_{S_{h,n}}(t) &= S_{h_{n-1}}(t) - S_{h_{n-2}}(t),\ \Psi_{V_{h,n}}(t) = V_{h_{n-1}}(t) - V_{h_{n-2}}(t),\ \Psi_{E_{h,n}}(t) = E_{h_{n-1}}(t) - E_{h_{n-2}}(t), \\ \Psi_{I_{h,n}}(t) &= I_{h_{n-1}}(t) - I_{h_{n-2}}(t),\ \Psi_{R_{h,n}}(t) = R_{h_{n-1}}(t) - R_{h_{n-2}}(t),\ \Psi_{L_n}(t) = L_{n-1}(t) - L_{n-2}(t), \\ \Psi_{S_{r,n}}(t) &= S_{r_{n-1}}(t) - S_{r_{n-2}}(t),\ \Psi_{E_{r,n}}(t) = E_{r_{n-1}}(t) - E_{r_{n-2}}(t),\ \Psi_{I_{r,n}}(t) = I_{r_{n-1}}(t) - I_{r_{n-2}}(t), \end{aligned} \tag{5.10}$$

we obtain the following

$$\begin{cases} \|\Psi_{S_{h,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_1 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{S_{h,n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{V_{h,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_2 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{V_{h,n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{E_{h,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_3 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{E_{h,n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{I_{h,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_4 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{I_{h,n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{R_{h,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_5 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{R_{h,n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{L_n}(t)\| \leq \mathbb{H}(\alpha)\zeta_6 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{L_{n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{S_{r,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_7 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{S_{r,n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{E_{r,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_8 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{E_{r,n-1}}(\Theta)\| d\Theta, \\ \|\Psi_{I_{r,n}}(t)\| \leq \mathbb{H}(\alpha)\zeta_9 \int_0^t (t-\Theta)^{-\alpha} \|\Psi_{I_{r,n-1}}(\Theta)\| d\Theta. \end{cases} \tag{5.11}$$

To conclude the existence and uniqueness of our proposed Caputo fractional model, we need to prove Theorem 7.

Theorem 7. Supposing that the following conditions satisfies

$$\frac{\mathbb{H}(\alpha)}{(\alpha)} K^{\alpha} \zeta_i < 1, \quad i = 1, 2, 3, ..., 9.$$

Then our proposed Caputo fractional model has unique solution for all $t \in [0, K]$.

Proof. We have established that $S_h(t),\ V_h(t),\ E_h(t), I_h(t),\ R_h(t),\ L(t),\ S_v(t),\ E_v(t)$, and $I_v(t)$ are upper bound. Also from (5.5) and (5.6) as well as Υ_i where $i = 1, 2, 3, ..., 9$ satisfies Lipschitz condition, then, using (5.11) through the recursive principle, we arrive at

$$\begin{cases} \|\Psi_{S_{h,n}}(t)\| \leq \|S_{h_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_1\right)^n, \\ \|\Psi_{V_{h,n}}(t)\| \leq \|V_{h_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_2\right)^n, \\ \|\Psi_{E_{h,n}}(t)\| \leq \|E_{h_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_3\right)^n, \\ \|\Psi_{I_{h,n}}(t)\| \leq \|I_{h_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_4\right)^n, \\ \|\Psi_{R_{h,n}}(t)\| \leq \|R_{h_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_5\right)^n, \\ \|\Psi_{L_n}(t)\| \leq \|L_0(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_6\right)^n, \\ \|\Psi_{S_{r,n}}(t)\| \leq \|S_{r_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_7\right)^n, \\ \|\Psi_{E_{r,n}}(t)\| \leq \|E_{r_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_8\right)^n, \\ \|\Psi_{I_{r,n}}(t)\| \leq \|I_{r_0}(t)\| \left(\frac{\mathbb{H}(\alpha)}{\alpha} K^{\alpha} \zeta_9\right)^n. \end{cases} \tag{5.12}$$

By limit principle as $n \to \infty$, the sequence hold and exist for $\|\Psi_{S_{h,n}}(t)\| \to 0$, $\|\Psi_{V_{h,n}}(t)\| \to 0$, $\|\Psi_{E_{h,n}}(t)\| \to 0$, $\|\Psi_{I_{h,n}}(t)\| \to 0$, $\|\Psi_{R_{h,n}}(t)\| \to 0$, $\|\Psi_{L_n}(t)\| \to 0$, $\|\Psi_{S_{r,n}}(t)\| \to 0$, $\|\Psi_{E_{r,n}}(t)\| \to 0$, $\|\Psi_{I_{r,n}}(t)\| \to 0$. Also applying the triangle inequality with (5.12), for any m, we obtain

$$\begin{cases} \|S_{h_{n+m}}(t) - S_{h_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_1^j = \frac{y_1^{n+1}-y_1^{n+m+1}}{1-y_1}, \\ \|V_{h_{n+m}}(t) - V_{h_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_2^j = \frac{y_2^{n+1}-y_2^{n+m+1}}{1-y_2}, \\ \|E_{h_{n+m}}(t) - E_{h_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_3^j = \frac{y_3^{n+1}-y_3^{n+m+1}}{1-y_3}, \\ \|I_{h_{n+m}}(t) - I_{h_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_4^j = \frac{y_4^{n+1}-y_4^{n+m+1}}{1-y_4}, \\ \|R_{h_{n+m}}(t) - R_{h_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_5^j = \frac{y_5^{n+1}-y_5^{n+m+1}}{1-y_5}, \\ \|L_{n+m}(t) - L_n(t)\| \leq \sum_{j=n+1}^{n+m} y_6^j = \frac{y_6^{n+1}-y_6^{n+m+1}}{1-y_6}, \\ \|S_{r_{n+m}}(t) - S_{r_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_7^j = \frac{y_7^{n+1}-y_7^{n+m+1}}{1-y_7}, \\ \|E_{r_{n+m}}(t) - E_{r_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_8^j = \frac{y_8^{n+1}-y_8^{n+m+1}}{1-y_8}, \\ \|I_{r_{n+m}}(t) - I_{r_n}(t)\| \leq \sum_{j=n+1}^{n+m} y_9^j = \frac{y_9^{n+1}-y_9^{n+m+1}}{1-y_9}, \end{cases} \tag{5.13}$$

where $y_i = \frac{\mathbb{H}(\alpha)}{\alpha} K^\alpha \zeta_i < 1, \quad i = 1, 2, 3, ..., 9$. by the assumption. Therefore, $S_{h_n}, V_{h_n}, E_{h_n}, I_{h_n}, R_{h_n}, L_n, S_{r_n}, E_{r_n}, I_{r_n}$ are Cauchy sequences in the Banach space $\mathtt{W(J)}$. Therefore, the state variables converges uniformly. Hence, imposing the limit theorem in (5.7) as $n \to \infty$ affirms that the limit of this sequence is the unique solution of our proposed Caputo fractional model. Hence, we conclude that the existence of the unique solution of our proposed Caputo fractional model has been proved.

6. Numerical schemes

Here, using two-step Lagrange interpolation, we present the numerical algorithms for the Caputo fractional epidemiological model for the MPXV that takes vaccination and environmental transmission.For more details about the numerical analysis see [40]. The formula for the Cauchy problem of the Caputo derivative is

$$\begin{aligned} {}_0^{\mathrm{C}}D_t^\alpha u(t) &= \Upsilon(t, u(t)), \\ u(0) &= u_0. \end{aligned} \tag{6.1}$$

With the help of the Caputo integral, (6.1) can be transformed into

$$u(t) - u(0) = \frac{1}{\Gamma(\alpha)} \int_0^t \Upsilon(\Theta, u(\Theta))(t - \Theta)^{\alpha-1} d\Theta. \tag{6.2}$$

At the point $t_{w+1} = (w + 1)h$ and $t_w = wh$, $w = 0, 1, 2, 3, 4, ...$ with h being the time step, Eq (6.2) can be formulated as:

$$u(t_{w+1}) - u(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{w+1}} \Upsilon(\Theta, u(\Theta))(t_{w+1} - \Theta)^{\alpha-1} d\Theta. \tag{6.3}$$

Which can be written as

$$u(t_{w+1}) = u(0) + \frac{1}{\Gamma(\alpha)} \sum_{q=0}^{w} \int_{t_q}^{t_{q+1}} \Upsilon(\Theta, u(\Theta))(t_{w+1} - \Theta)^{\alpha-1} d\Theta. \tag{6.4}$$

We further breakdown the right hand side of Eq (6.4) and using the Lagrange polynomial, Eq (6.4) can now be written as

$$u_{w+1} = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{q=0}^{w} \left\{ \begin{array}{l} \frac{\Upsilon(t_q,u_q)}{h} \int_{t_q}^{t_{q+1}} (\Theta - t_{q-1})(t_{w+1} - \Theta)^{\alpha-1} d\Theta \\ -\frac{\Upsilon(t_{q-1},u_{q-1})}{h} \int_{t_q}^{t_{q+1}} (\Theta - t_q)(t_{w+1} - \Theta)^{\alpha-1} d\Theta \end{array} \right\}. \tag{6.5}$$

Where Eq (6.5) can be written as

$$\begin{aligned} u_{w+1} = {} & u_0 + \frac{1}{\Gamma(\alpha)} \sum_{q=0}^{w} \frac{\Upsilon(t_q, u_q)}{h} \int_{t_q}^{t_{q+1}} (\Theta - t_{q-1})(t_{w+1} - \Theta)^{\alpha-1} d\Theta \\ & - \frac{1}{\Gamma(\alpha)} \sum_{q=0}^{w} \frac{\Upsilon(t_{q-1}, u_{q-1})}{h} \int_{t_q}^{t_{q+1}} (\Theta - t_q)(t_{w+1} - \Theta)^{\alpha-1} d\Theta. \end{aligned} \tag{6.6}$$

When we solve the integral in (6.6), we get the result that follows:

$$\int_{t_q}^{t_{q+1}} (\Theta - t_{q-1})(t_{w+1} - \Theta)^{\alpha-1} d\Theta = \frac{h^{\alpha-1}}{\alpha(\alpha+1)} \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix}, \tag{6.7}$$

$$\int_{t_q}^{t_{q+1}} (\Theta - t_q)(t_{w+1} - \Theta)^{\alpha-1} d\Theta = \frac{h^{\alpha-1}}{\alpha(\alpha+1)} \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+\alpha) \end{bmatrix}. \tag{6.8}$$

Replacing Eqs (6.7) and (6.8) into Eq (6.6), the numerical algorithm for the Caputo derivative is as follows

$$\begin{aligned} u_{w+1} = {} & u_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, u_q) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\ & - \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, u_{q-1}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+\alpha) \end{bmatrix}. \end{aligned} \tag{6.9}$$

Thus, in terms of our Caputo fractional epidemiological model for the MPXV, we get:

$$\begin{aligned} S_{h_{w+1}} = {} & S_{h_0} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, S_{h_q}) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\ & - \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, S_{h_{q-1}}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}, \\ V_{h_{w+1}} = {} & V_{h_0} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, V_{h_q}) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\ & - \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, V_{h_{q-1}}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}, \\ E_{h_{w+1}} = {} & E_{h_0} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, E_{h_q}) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \end{aligned}$$

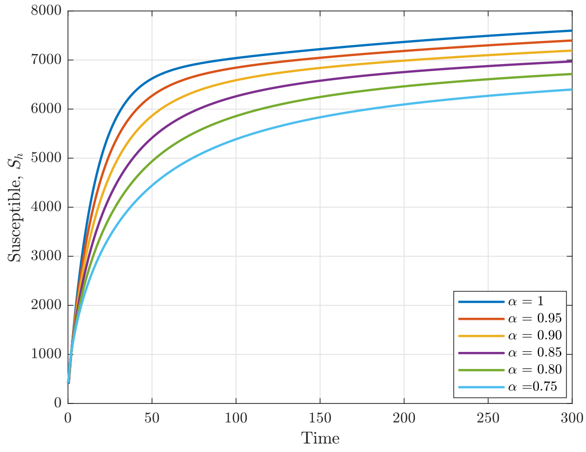
$$
\begin{aligned}
&- \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, S_{h_{q-1}}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}, \\
I_{h_{w+1}} = I_{h_0} &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, I_{h_q}) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\
&- \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, I_{h_{q-1}}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}, \\
R_{h_{w+1}} = R_{h_0} &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, R_{h_q}) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\
&- \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, R_{h_{q-1}}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}, \\
L_{w+1} = L_0 &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, L_q) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\
&- \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, L_{q-1}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}, \\
S_{r_{w+1}} = S_{r_0} &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, S_{r_q}) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\
&- \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, S_{r_{q-1}}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}, \\
I_{r_{w+1}} = I_{r_0} &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_q, I_{r_q}) \begin{bmatrix} (w-q+1)^{\alpha}(w-q+2+\alpha) \\ -(w-q)^{\alpha}(w-q+2+2\alpha) \end{bmatrix} \\
&- \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{q=0}^{w} \Upsilon(t_{q-1}, I_{r_{q-1}}) \begin{bmatrix} (w-q+1)^{\alpha+1} \\ -(w-q)^{\alpha}(w-q+1+q) \end{bmatrix}.
\end{aligned} \tag{6.10}
$$

7. Numerical simulation and discussion

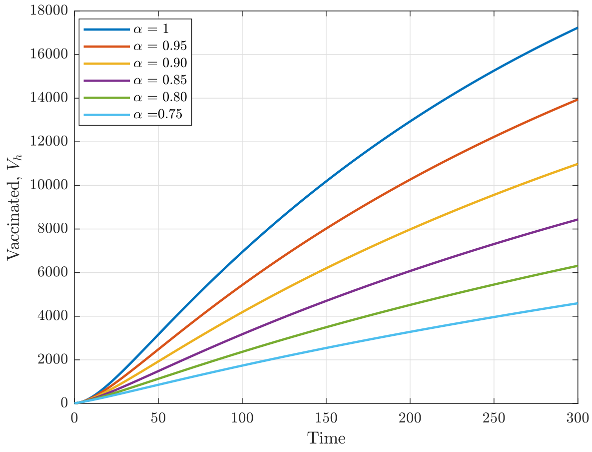
The physical outlook of our proposed model under the Caputo fractional operator is depicted in Figures 1 and 2. Using the Adams-Bashforth method, taken account of the following initial conditions; $S_h(0) = 500, V_h(0) = 300, E_h(0) = 350, I_h(0) = 200, R_h(0) = 250, L(0) = 200, S_r(0) = 500, E_r(0) = 400, I_r(0) = 220$. From Figures 1 and 2 we observed in mathematical sense that the fractional order is proportional to the MPXV transmission, thus when the operator α is varied the dynamism of the virus changes. For example, when the fractional order α is lowered from 1 in Figure 1(a),(b) it captured low susceptible and vaccination compartment. In biological sense, people have less health education, the rate of vaccination and vaccine efficacy of the MPXV is low. In Figure 1(c),(d), we see high number of people exposed and infected due to the Figure 1(a),(b) influence. In Figures 3 and 4 we maintain the fractional operator to be fixed at $\alpha = 0.95$ and varied some of the sensitive parameters which are significant and need much attention. From Figure 3(a)–(f) we observed that when we increase the rate of vaccine and efficacy rate, people are motivated to receive vaccine which reduce infection rate. In Figure 4(a),(b), when we increase recovery rate through practice such as offering fluids and food to maintain adequate nutritional status, infection decrease and minimal recovery. In Figure 4(c)–(e), it is easy to see the numerical trajectory of exposed, infection and recovery, when the parameter γ_h varied. In Figure 4(f), the environmental transmission has been observed to be one of the main transmission for the MPOX infection. When we simulate the impact of m_1 and m_2 with different rates, we noticed that that the environmental transmission reduces. According to this understanding, the prevalence of disease can be decreased by practising good personal hygiene and using appropriate disinfection sprays to remove viruses from the environment.

8. Conclusions

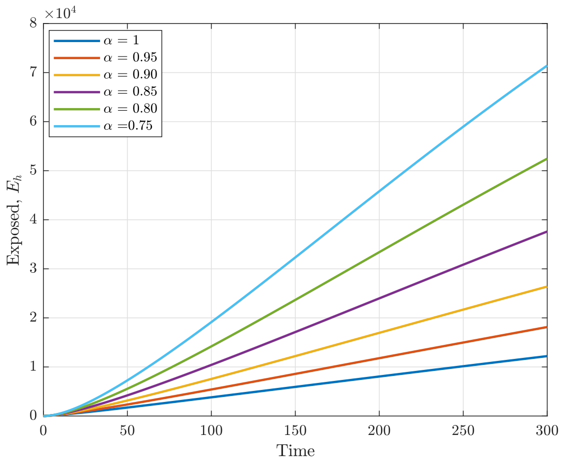
We have comprehensively analyzed the MPXV transmission dynamics under Caputo fractional order derivative in this paper. We have investigated the qualitative aspect of the spread of MPXV by analysing the invariant, positiveness, boundedness, equilibrium point, and fundamental reproductive number $\Re$. We have also examined local and global stability of our proposed model. The solution of our proposed model existence and uniqueness are also examined using fixed point theorem. Numerical trajectories are obtained for nine compartments in the fractional order model. Furthermore, we explored some of the sensitive parameters impact. Based on the trajectories, we hypothesised that the memory index or fractional order could use to control the MPXV transmission dynamics. It is also seen that if the proper vaccination is administrated and practice such as personal hygiene and proper disinfection spray, the infected individuals decreases. We think that the research presented in the study will help the community's health and decision-making authorities fight the disease. Future versions of the model could be created by combining appropriate time-dependent control actions with real data.



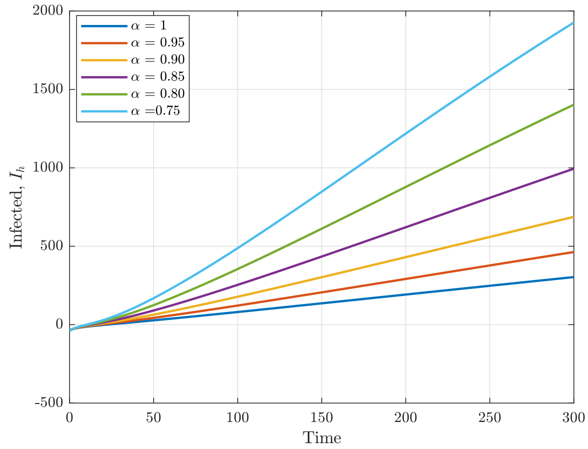
(a) Dynamics of Susceptible Class of humans



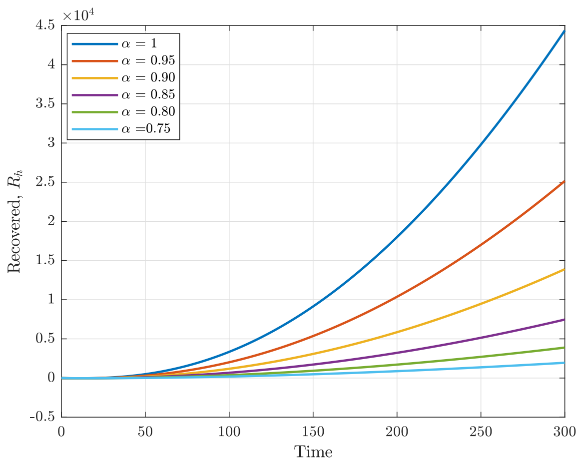
(b) Dynamics of Vaccination Class of humans



(c) Dynamics of Exposed class of humans

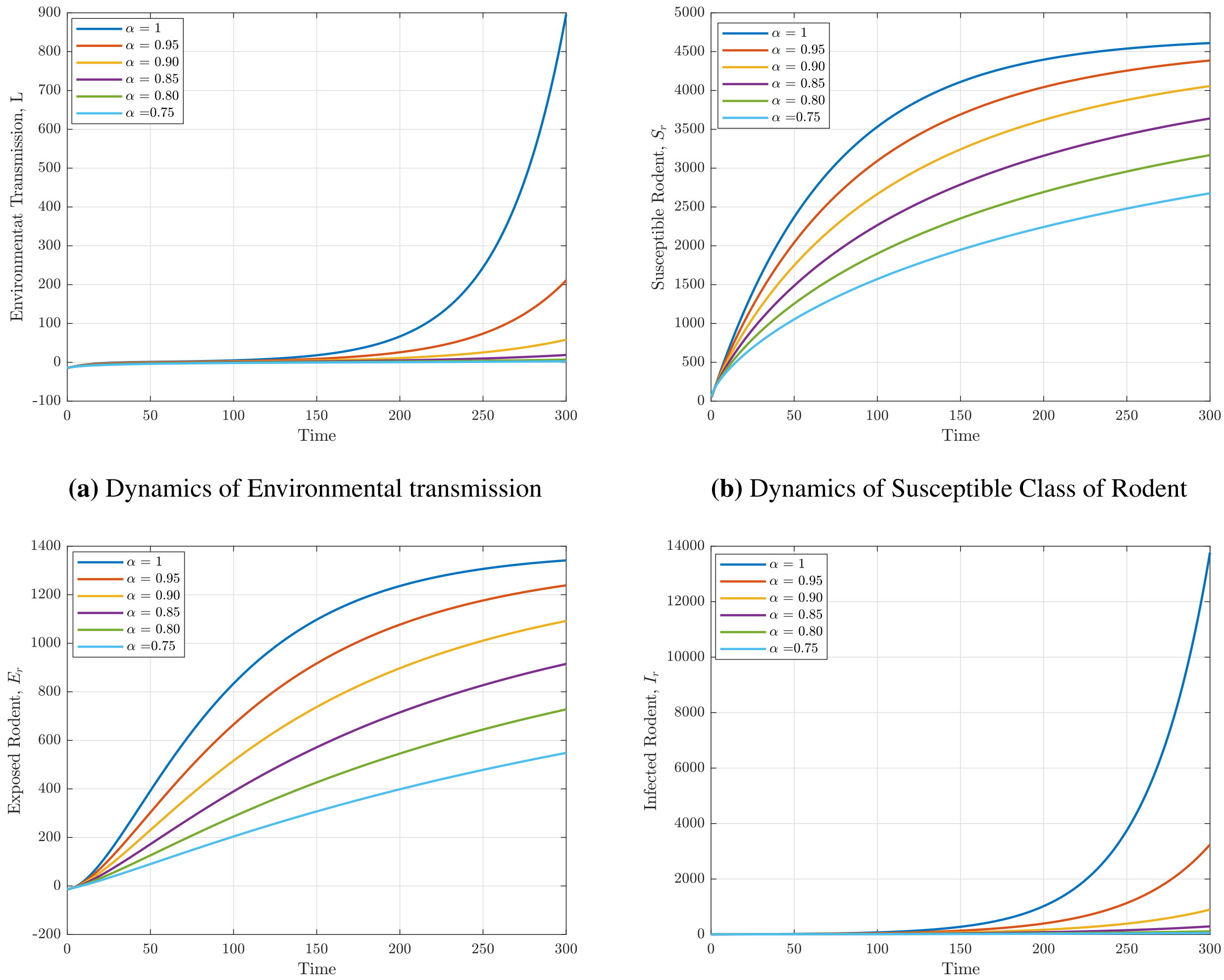


(d) Dynamics of Infected Class of humans



(e) Dynamics of Recovery Class of humans

Figure 1. Numerical trajectory of MPXV transmission under Caputo fractional operator.



(a) Dynamics of Environmental transmission

(b) Dynamics of Susceptible Class of Rodent

(c) Dynamics of Exposed class of Rodent

(d) Dynamics of Infected Class of Rodent

Figure 2. Numerical trajectory of MPXV transmission under Caputo fractional operator.

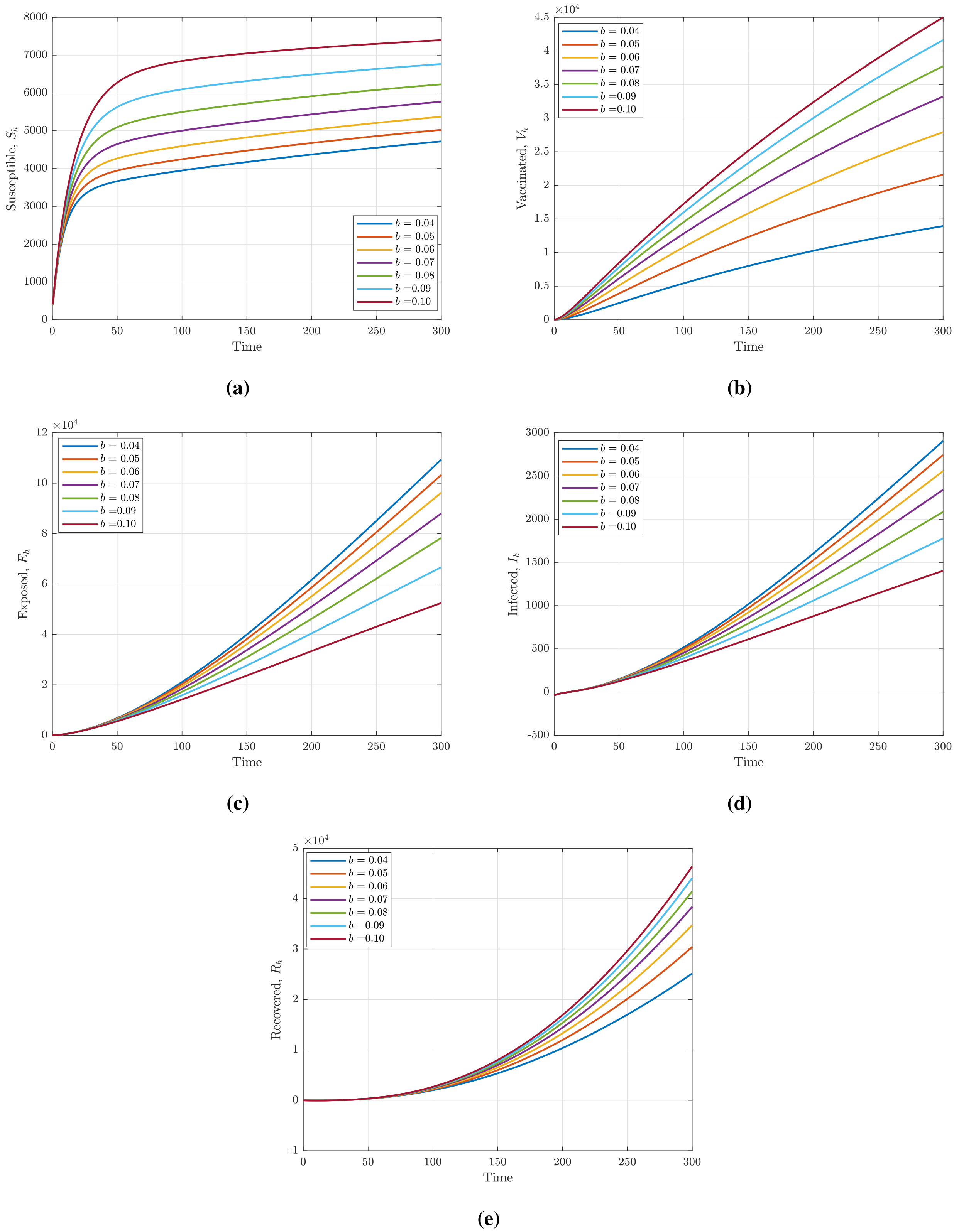


Figure 3. Numerical trajectory when one varying b against fractional operator $\alpha = 0.95$.

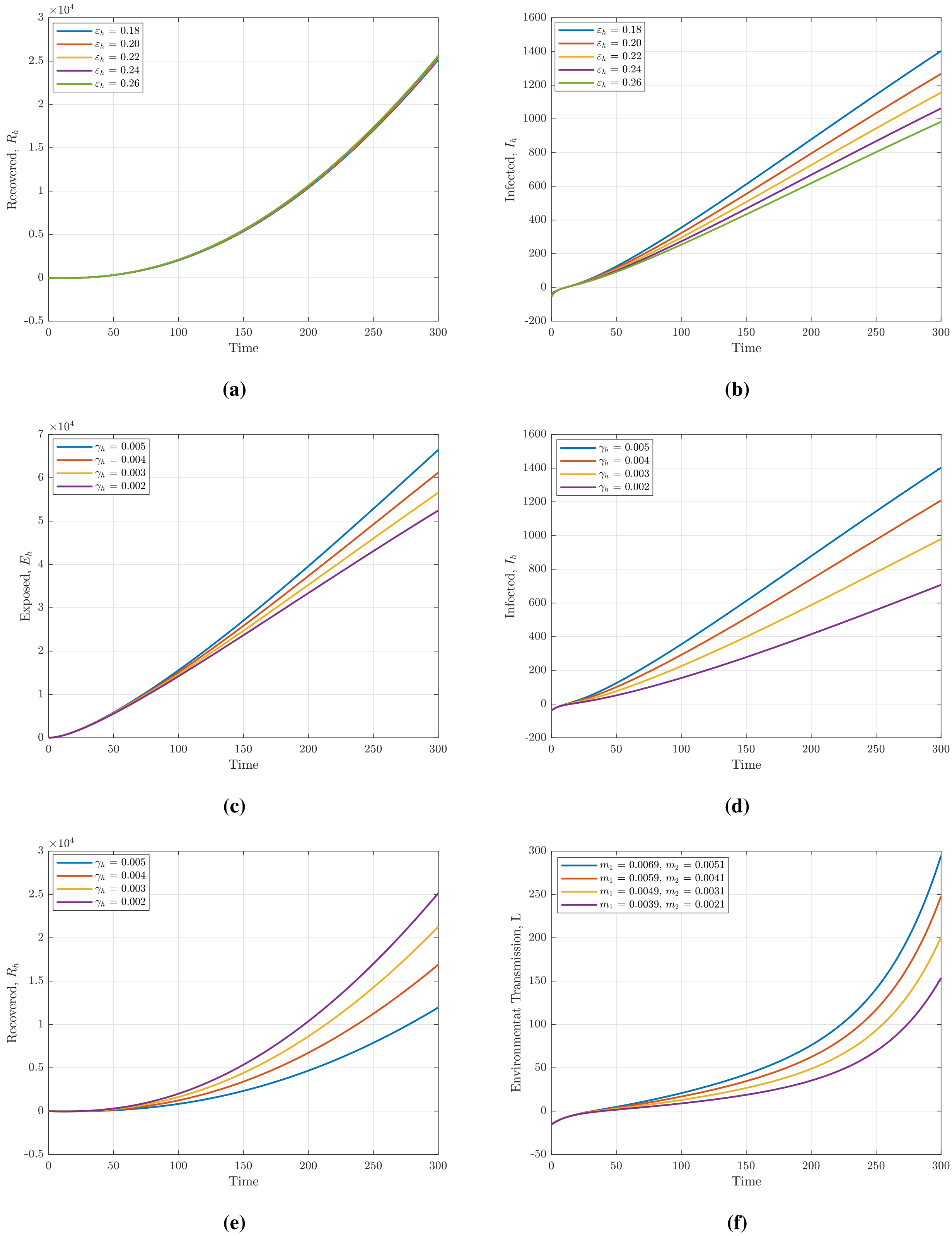


Figure 4. Numerical trajectory when one varying (ε_h, γ_h, m) against fractional operator $\alpha = 0.95$.

Acknowledgements

Authors would like to appreciate support from Human Sciences Research Council (HSRC) for giving us the platform to collaborate for this work.

Conflict of interest

The authors state that they do not have any competing interests.

Authors contribution

All authors contributed and supported the writing of this manuscript equally, and the final paper was read and approved.

References

1. *World Health Organization (WHO)*, 2022 Mpox (Monkeypox) outbreak: global trends, 2022. Available from: https://worldhealthorg.shinyapps.io/mpx_global/.
2. H. Harapan, Y. Ophinni, D. Megawati, A. Frediansyah, S. S. Mamada, M. Salampe, et al., Monkeypox: a comprehensive review, *Viruses*, **14** (2022), 2155. https://doi.org/10.3390/v14102155
3. E. M. Bunge, B. Hoet, L. Chen, F. Lienert, H. Weidenthaler, L. R. Baer, et al., The changing epidemiology of human monkeypox-A potential threat? A systematic review, *PLoS Negl. Trop. Dis.*, **16** (2022), e0010141. https://doi.org/10.1371/journal.pntd.0010141
4. *Centers for Disease Control and Prevention (CDC)*, 2022 Mpox outbreak global map, 2022. Available from: https://www.cdc.gov/poxvirus/monkeypox/response/2022/world-map.html.
5. *World Health Organization (WHO)*, WHO director-general declares the ongoing Monkeypox otbreak a public health emergency of international concern, 2022. Available from: https://www.who.int/europe/news/item/23-07-2022-who-director-general-declares-the-ongoing-monkeypox-outbreak-a-public-health-event-of-international-concern.
6. N. I. O. Silva, J. S. de Oliveira, E. G. Kroon, G. de Souza Trindade, B. P. Drumond, There, and everywhere: The wide host range and geographic distribution of zoonotic orthopoxviruses, *Viruses*, **13** (2020), 43. https://doi.org/10.3390/v13010043
7. Y. Huang, L. Mu, W. Wang, Monkeypox: Epidemiology, pathogenesis, treatment and prevention, *Signal Transduct. Target. Ther.*, **7** (2022), 373. https://doi.org/10.1038/s41392-022-01215-4
8. M. G. Reynolds, D. S. Carroll, K. L. Karem, Factors affecting the likelihood of monkeypox's emergence and spread in the post-smallpox era, *Curr. Opin. Virol.*, **2** (2012), 335–343. https://doi.org/10.1016/j.coviro.2012.02.004
9. S. Usman, I. I. Adamu, Modeling the transmission dynamics of the Monkeypox virus infection with treatment and vaccination interventions, *J. Appl. Math. Phys.*, **5** (2017), 2335–2353. https://doi.org/10.4236/jamp.2017.512191

10. A. B. Al-Tammemi, R. Albakri, S. Alabsi, The outbreak of human Monkeypox in 2022: A changing epidemiology or an impending aftereffect of smallpox eradication? *Front. Trop. Dis.*, **3** (2022), 951380. https://doi.org/10.3389/fitd.2022.951380

11. L. G. Lulli, A. Baldassarre, N. Mucci, G. Arcangeli, Prevention, risk exposure, and knowledge of Monkeypox in occupational settings: a scoping review, *Trop. Med. Infect. Dis.*, **7** (2022), 276. https://doi.org/10.3390/tropicalmed7100276

12. S. Parker, A. Nuara, R. M. Buller, D. A. Schultz, Human monkeypox: an emerging zoonotic disease, *Future Microbiol.*, **2** (2007), 17–34. https://doi.org/10.2217/17460913.2.1.17

13. K. Brown, P. A. Leggat, Human monkeypox: Current state of knowledge and implications for the future, *Trop. Med. Infect. Dis.*, **1** (2016), 8. https://doi.org/10.3390/tropicalmed1010008

14. *World Health Organization (WHO)*, Monkeypox strategic preparedness, readiness, and response plan (SPRP), 2022. Available from: https://www.who.int/publications/m/item/monkeypox-strategic-preparedness–readiness–and-response-plan-(sprp).

15. N. Z. Alshahrani, F. Alzahrani, A. M. Alarifi, M. R. Algethami, M. N. Alhumam, H. A. M. Ayied, et al., Assessment of knowledge of monkeypox viral infection among the general population in Saudi Arabia, *Pathogens*, **11** (2022), 904. https://doi.org/10.3390/pathogens11080904

16. H. Harapan, A. M. Setiawan, A. Yufika, S. Anwar, S. Wahyuni, F. W. Asrizal, et al., Knowledge of human monkeypox viral infection among general practitioners: a cross-sectional study in Indonesia, *Pathog. Glob. Health*, **114** (2020), 68–75. https://doi.org/10.1080/20477724.2020.1743037

17. M. Sallam, K. Al-Mahzoum, L. A. Dardas, A. B. Al-Tammemi, L. Al-Majali, H. Al-Naimat, et al., Knowledge of human monkeypox and its relation to conspiracy beliefs among students in Jordanian health schools: filling the knowledge gap on emerging zoonotic viruses, *Medicina*, **58** (2022), 924. https://doi.org/10.3390/medicina58070924

18. A. Kaur, R. Goel, R. Singh, A. Bhardwaj, R. Kumari, R. S. Gambhir, Identifying monkeypox: do dental professionals have adequate knowledge and awareness, *Rocz. Panstw. Zakl. Hig.*, **73** (2022), 365–371. https://doi.org/10.32394/rpzh.2022.0226

19. H. Harapan, A. L. Wagner, A. Yufika, A. M. Setiawan, S. Anwar, S. Wahyuni, et al., Acceptance and willingness to pay for a hypothetical vaccine against monkeypox viral infection among frontline physicians: a cross-sectional study in Indonesia, *Vaccine*, **38** (2020), 6800–6806. https://doi.org/10.1016/j.vaccine.2020.08.034

20. C. Dong, Z. Yu, Y. Zhao, X. Ma, Knowledge and vaccination intention of monkeypox in China's general population: a cross-sectional online survey, *Travel Med. Infect. Dis.*, **52** (2023), 102533. https://doi.org/10.1016/j.tmaid.2022.102533

21. E. Addai, L. Zhang, J. K. K. Asamoah, J. F. Essel, A fractional order age-specific smoke epidemic model, *Appl. Math. Modell.*, **119** (2023), 99–118. https://doi.org/10.1016/j.apm.2023.02.019

22. M. Higazy, M. A. Alyami, New Caputo-Fabrizio fractional order SEIASqEqHR model for COVID-19 epidemic transmission with genetic algorithm based control strategy, *Alex. Eng. J.*, **59** (2020), 4719–4736. https://doi.org/10.1016/j.aej.2020.08.034

23. D. Baleanu, A. Jajarmi, H. Mohammad, S. Rezapour, A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative, *Chaos, Solitons Fractals*, **134** (2020), 109705. https://doi.org/10.1016/j.chaos.2020.109705

24. S. Wutiphol, A. Turab, Mathematical analysis of an extended SEIR model of COVID-19 using the ABC-Fractional operator, *Math. Comput. Simul.*, **198** (2022), 65–84. https://doi.org/10.1016/j.matcom.2022.02.009

25. J. K. K. Asamoah, E. Okyere, E. Yankson, A. A. Opoku, A. Adom-Konadu, E. Acheampong, et al., Non-fractional and fractional mathematical analysis and simulations for Q fever, *Chaos, Solitons Fractals*, **156** (2022), 111821. https://doi.org/10.1016/j.chaos.2022.111821

26. L. L. Zhang, E. Addai, J. Ackora-Prah, Y. D. Arthur, J. K. K. Asamoah, Fractional-order Ebola-Malaria coinfection model with a focus on detection and treatment rate, *Comput. Math. Methods Med.*, **2022** (2022), 6502598. https://doi.org/10.1155/2022/6502598

27. S. Kumar, R. P. Chauhan, S. Momani, S. Hadid, Numerical investigations on COVID-19 model through singular and non-singular fractional operators, *Numer. Methods Partial Differ. Equations*, **2020** (2020), 1–27. https://doi.org/10.1002/num.22707

28. M. Aslam, R. Murtaza, T. Abdeljawad, G. ur Rahman, A. Khan, H. Khan, et al., A fractional order HIV/AIDS epidemic model with Mittag-Leffler kernel, *Adv. Differ. Equations*, **107** (2021), 1–5. https://doi.org/10.1186/s13662-021-03264-5

29. O. J. Peter, F. A. Oguntolu, M. M. Ojo, A. O. Oyeniyi, R. Jan, I. Khan, Fractional order mathematical model of monkeypox transmission dynamics, *Phys. Scr.*, **97** (2022), 084005. https://doi.org/10.1088/1402-4896/ac7ebc

30. O. J. Peter, S. Kumar, N. Kumari, F. A. Oguntolu, K. Oshinubi, R. Musa, Transmission dynamics of Monkeypox virus: a mathematical modelling approach, *Model. Earth Syst. Environ.*, **8** (2022), 3423–3434. https://doi.org/10.1007/s40808-021-01313-2

31. M. Ngungu, E. Addai, A. Adeniji, U. M. Adam, K. Oshinubi K., Mathematical epidemiological modeling and analysis of monkeypox dynamism with non-pharmaceutical intervention using real data from United Kingdom, *Front. Public Health*, **11** (2023), 1101436. https://doi.org/10.3389/fpubh.2023.1101436

32. J. Singh, D. Kumar, M. A. Qurashi, D. Baleanu, A new fractional model for giving up smoking dynamics, *Adv. Differ. Equations*, **88** (2017). https://doi.org/10.1186/s13662-017-1139-9

33. Q. T. Ain, N. Anjum, D. Anwarud, A. Zeb, S. Djilali, Z. A Khan, On the analysis of Caputo fractional order dynamics of Middle East Lungs Coronavirus (MERS-CoV) model, *Alexandria Eng. J.*, **61** (2022), 5123–5131. https://doi.org/10.1016/j.aej.2021.10.016

34. U. T. Mustapha, S. Qureshi, A. Yusuf, E. Hincal, Fractional modeling for the spread of Hookworm infection under Caputo operator, *Chaos, Solitons Fractals*, **137** (2020), 109878. https://doi.org/10.1016/j.chaos.2020.109878

35. I. Ahmed, I. A. Baba, A. Yusuf, P. Kumam, W. Kumam, Analysis of Caputo fractional-order model for COVID-19 with lockdown, *Adv. Differ. Equations*, **394** (2020). https://doi.org/10.1186/s13662-020-02853-0

36. E. Addai, L. L. Zhang, A. K. Preko, J. K. K. Asamoah, Fractional order epidemiological model of SARS-CoV-2 dynamism involving Alzheimer's disease, *Healthcare Anal.*, **2** (2022), 100114. https://doi.org/10.1016/j.health.2022.100114

37. A. Abdulhamid, N. Hussaini, Effects of quarantine on transmission dynamics of Lassa fever, *Bayero J. Pure Appl. Sci.*, **11** (2019), 397. https://doi.org/10.4314/bajopas.v11i1.64S

38. E. Ahmed, A. M. A. El-Sayed, H. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, *J. Math. Anal. Appl.*, **325** (2007), 542–553. https://doi.org/10.1016/j.jmaa.2006.01.087

39. P. van den Driessche, J. Wanmough, Reproduction numbers and sub-threshold endemic equilibria for compartimental models of disease transmition, *Math. Biosci.*, **180** (2000), 29–48. https://doi.org/10.1016/s0025-5564(02)00108-6

40. A. Atangana, K. M. Owolabi, New numerical approach for fractional differential equations, *Math. Model. Nat. Phenom.*, **13** (2018), 3. https://doi.org/10.1051/mmnp/2018041